2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 10 - F.E. large deformation/general nonlinear analysis

Prof. K.J. Bathe MIT OpenCourseWare

We developed

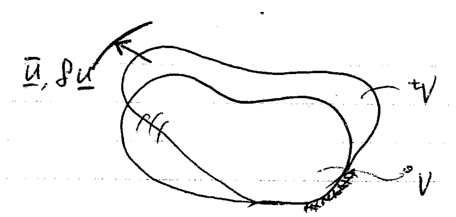
Reading: Ch. 6

$$\int_{^tV} {}^t\tau_{ij}\,{}_t\bar{e}_{ij}\,d^tV = {}^t\mathcal{R} \tag{10.1}$$

$$ {}_t\bar{e}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial\, {}^tx_j} + \frac{\partial \bar{u}_j}{\partial\, {}^tx_i}\right) \tag{10.2}$$

$$\int_{^tV} {}^t\tau_{ij}\,\delta_t e_{ij}\,d^tV = {}^t\mathcal{R} \tag{10.3}$$

$$\delta_t e_{ij} = \frac{1}{2}\left(\frac{\partial(\delta u_i)}{\partial\, {}^tx_j} + \frac{\partial(\delta u_j)}{\partial\, {}^tx_i}\right) \quad (\equiv\ {}_t\bar{e}_{ij}) \tag{10.4}$$



In FEA:

$$ {}^t\boldsymbol{F} = {}^t\boldsymbol{R} \tag{10.5}$$

In linear analysis

$$ {}^t\boldsymbol{F} = \boldsymbol{K}\,{}^t\boldsymbol{U} \Rightarrow \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \tag{10.6}$$

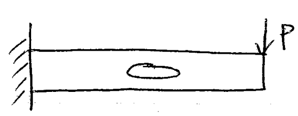
In general nonlinear analysis, we need to iterate. Assume the solution is known "at time t"

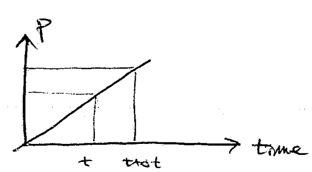
$$ {}^t\boldsymbol{x} = {}^0\boldsymbol{x} + {}^t\boldsymbol{u} \tag{10.7}$$

Hence ${}^t\boldsymbol{F}$ is known. Then we consider

$$ {}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R} \tag{10.8}$$

Consider the loads (applied external loads) to be deformation-independent, e.g.





Then we can write

$$ {}^{t+\Delta t}\boldsymbol{F} = {}^{t}\boldsymbol{F} + \boldsymbol{F} \tag{10.9} $$
$$ {}^{t+\Delta t}\boldsymbol{U} = {}^{t}\boldsymbol{U} + \boldsymbol{U} \tag{10.10} $$

where only ${}^{t}\boldsymbol{F}$ and ${}^{t}\boldsymbol{U}$ are known.

$$ \boldsymbol{F} \cong {}^{t}\boldsymbol{K}\Delta\boldsymbol{U}, \quad {}^{t}\boldsymbol{K} = \text{tangent stiffness matrix at time } t \tag{10.11} $$

From (10.8),

$$ {}^{t}\boldsymbol{K}\Delta\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t}\boldsymbol{F} \tag{10.12} $$

We use this to obtain an approximation to $\boldsymbol{U}$. We obtain a more accurate solution for $\boldsymbol{U}$ (i.e. ${}^{t+\Delta t}\boldsymbol{F}$) using

$$ {}^{t+\Delta t}\boldsymbol{K}^{(i-1)}\Delta\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)} \tag{10.13} $$
$$ {}^{t+\Delta t}\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{U}^{(i-1)} + \Delta\boldsymbol{U}^{(i)} \tag{10.14} $$

Also,

$$ {}^{t+\Delta t}\boldsymbol{F}^{(0)} = {}^{t}\boldsymbol{F} \tag{10.15} $$
$$ {}^{t+\Delta t}\boldsymbol{K}^{(0)} = {}^{t}\boldsymbol{K} \tag{10.16} $$
$$ {}^{t+\Delta t}\boldsymbol{U}^{(0)} = {}^{t}\boldsymbol{U} \tag{10.17} $$

Iterate for $i = 1, 2, 3\ldots$ until convergence. Convergence is reached when

$$ \left\|\Delta\boldsymbol{U}^{(i)}\right\|_2 < \epsilon_D \tag{10.18} $$
$$ \left\|{}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)}\right\|_2 < \epsilon_F \tag{10.19} $$

Note:

$$ \|\boldsymbol{a}\|_2 = \sqrt{\sum_i (a_i)^2} $$
$$ \sum_{i=1,2,3\ldots} \Delta\boldsymbol{U}^{(i)} = \boldsymbol{U} $$

$\Delta\boldsymbol{U}^{(1)}$ in (10.13) is $\Delta\boldsymbol{U}$ in (10.12).

(10.13) is the *full* Newton-Raphson iteration.

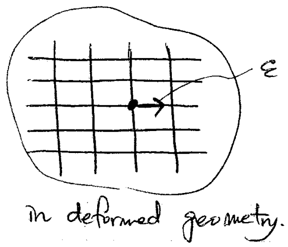
How we could (in principle) calculate ${}^{t}\boldsymbol{K}$

Process

- Increase the displacement ${}^{t}U_i$ by ϵ, with no increment for all ${}^{t}U_j$, $j \neq i$
- calculate ${}^{t+\epsilon}\boldsymbol{F}$
- the i-th column in ${}^{t}\boldsymbol{K} = \left({}^{t+\epsilon}\boldsymbol{F} - {}^{t}\boldsymbol{F}\right)/\epsilon = \frac{\partial\, {}^{t}\boldsymbol{F}}{\partial\, {}^{t}U_i}$.

So, perform this process for $i = 1, 2, 3, \ldots, n$, where n is the total number of degrees of freedom. Pictorially,

$$^t\boldsymbol{K} = \begin{bmatrix} \vdots & \vdots & \\ \vdots & \vdots & \cdots \\ \vdots & \vdots & \end{bmatrix}$$



A **general** difficulty: we cannot "simply" increment Cauchy stresses.

- $^{t+\Delta t}\tau_{ij}$ referred to area at time $t + \Delta t$
- $^t\tau_{ij}$ referred to area at time t.

We define a new stress measure, *2nd Piola - Kirchhoff stress*, $^{t+\Delta t}_{0}S_{ij}$, where 0 in the leading subscript refers to original configuration. Then,

$$^{t+\Delta t}_{0}S_{ij} = {}^t_0S_{ij} + {}_0S_{ij} \tag{10.20}$$

The strain measure energy-conjugate to the 2nd P-K stress $^t_0S_{ij}$ is the Green-Lagrange strain $^t_0\epsilon_{ij}$

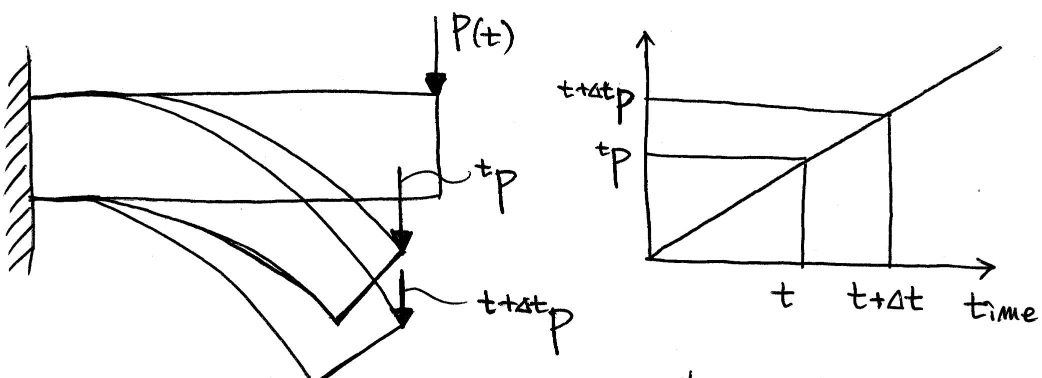
Then,

$$\int_{^0V} {}^t_0S_{ij}\ \delta\, {}^t_0\epsilon_{ij}\ d^0V = {}^t\mathcal{R} \tag{10.21}$$

Also,

$$\int_{^0V} {}^{t+\Delta t}_{0}S_{ij}\ \delta\, {}^{t+\Delta t}_{0}\epsilon_{ij}\ d^0V = {}^{t+\Delta t}\mathcal{R} \tag{10.22}$$

Example



$$^t\underline{F} = {}^t\underline{R}$$

$$ {}^{t}\boldsymbol{F} = {}^{t}\boldsymbol{R} \tag{10.23} $$

$$ {}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R} \tag{10.24} $$

(every time it is in equilibrium)

(10.13) and (10.14) give:

$i = 1$,

$$ {}^{t+\Delta t}\boldsymbol{K}^{(0)}\, \Delta\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(0)} \equiv \mathrm{fn}({}^{t}\boldsymbol{U}) \tag{10.25} $$

$$ {}^{t+\Delta t}\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{U}^{(0)} + \Delta\boldsymbol{U}^{(1)} \tag{10.26} $$

$i = 2$,

$$ {}^{t+\Delta t}\boldsymbol{K}^{(1)}\, \Delta\boldsymbol{U}^{(2)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(1)} \tag{10.27} $$

$$ {}^{t+\Delta t}\boldsymbol{U}^{(2)} = {}^{t+\Delta t}\boldsymbol{U}^{(1)} + \Delta\boldsymbol{U}^{(2)} \tag{10.28} $$