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2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

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Lecture 12 - Total Lagrangian formulation

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We discussed:

$${}^t\mathbf{X} = \begin{bmatrix} \frac{\partial^t x_i}{\partial^0 x_j} \end{bmatrix} \Rightarrow d^t \mathbf{x} = {}^t\mathbf{X} d^0 \mathbf{x}, \quad d^0 \mathbf{x} = ({}^t\mathbf{X})^{-1} d^t \mathbf{x} \quad (12.1)$$

$${}^t\mathbf{C} = {}^t\mathbf{X}^T {}^t\mathbf{X} \quad (12.2)$$

$$d^0 \mathbf{x} = {}^0\mathbf{X} d^t \mathbf{x} \quad \text{where } {}^0\mathbf{X} = ({}^t\mathbf{X})^{-1} = \begin{bmatrix} \frac{\partial^0 x_i}{\partial^t x_j} \end{bmatrix} \quad (12.3)$$

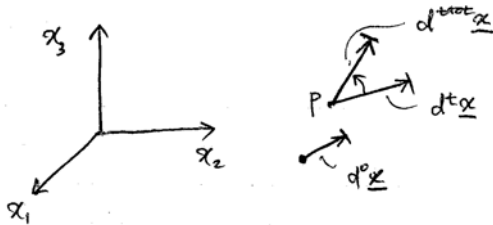
The Green-Lagrange strain:

$${}^t\boldsymbol{\epsilon} = \frac{1}{2} ({}^t\mathbf{X}^T {}^t\mathbf{X} - \mathbf{I}) = \frac{1}{2} ({}^t\mathbf{C} - \mathbf{I}) \quad (12.4)$$

Polar decomposition:

$${}^t\mathbf{X} = {}^t\mathbf{R} {}^t\mathbf{U} \Rightarrow {}^t\boldsymbol{\epsilon} = \frac{1}{2} (({}^t\mathbf{U})^2 - \mathbf{I}) \quad (12.5)$$

We see, physically that:



where $d^{t+\Delta t} \mathbf{x}$ and $d^t \mathbf{x}$ are the same lengths
 \Rightarrow the components of the G-L strain do not change.

Note in FEA

$$\left. \begin{aligned} {}^0 x_i &= \sum_k h_k {}^0 x_i^k \\ {}^t u_i &= \sum_k h_k {}^t u_i^k \end{aligned} \right\} \text{for an element} \quad (12.6)$$

$${}^t x_i = {}^0 x_i + {}^t u_i \rightarrow \text{for any particle} \quad (12.7)$$

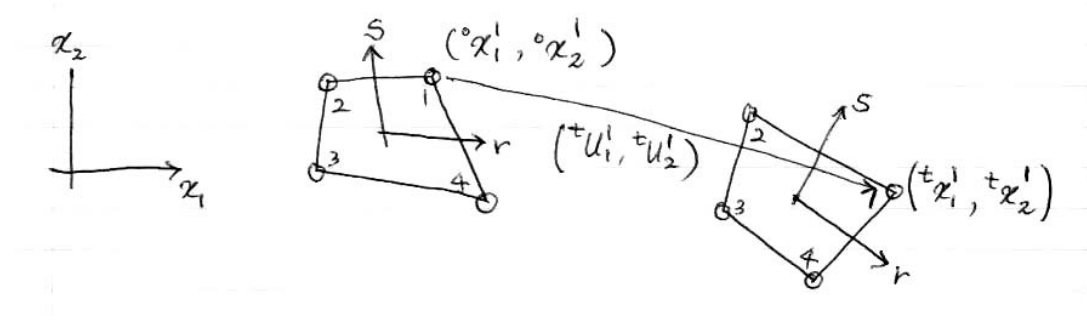
Hence for the element

$${}^t x_i = \sum_k h_k {}^0 x_i^k + \sum_k h_k {}^t u_i^k \quad (12.8)$$

$$= \sum_k h_k ({}^0 x_i^k + {}^t u_i^k) \quad (12.9)$$

$$= \sum_k h_k {}^t x_i^k \quad (12.10)$$

E.g., $k = 4$



2nd Piola-Kirchhoff stress

$${}^0_t\mathbf{S} = \frac{{}^0\rho}{{}^t\rho} {}^0\mathbf{X} {}^t\boldsymbol{\tau} {}^0\mathbf{X}^T \rightarrow \text{components also independent of a rigid body rotation} \quad (12.11)$$

Then

$$\int_{0V} {}^tS_{ij} \delta_0^t \epsilon_{ij} d^0V = \int_{tV} {}^t\tau_{ij} \delta_t e_{ij} d^tV = {}^t\mathcal{R} \quad (12.12)$$

We can use an incremental decomposition of stress/strain.

$${}^{t+\Delta t}{}_0\mathbf{S} = {}^t{}_0\mathbf{S} + {}_0\mathbf{S} \quad (12.13)$$

$${}^{t+\Delta t}{}_0S_{ij} = {}^t{}_0S_{ij} + {}_0S_{ij} \quad (12.14)$$

$${}^{t+\Delta t}{}_0\boldsymbol{\epsilon} = {}^t{}_0\boldsymbol{\epsilon} + {}_0\boldsymbol{\epsilon} \quad (12.15)$$

$${}^{t+\Delta t}{}_0\epsilon_{ij} = {}^t{}_0\epsilon_{ij} + {}_0\epsilon_{ij} \quad (12.16)$$

Assume the solution is known at time t , calculate the solution at time $t + \Delta t$. Hence, we apply (12.12) at time $t + \Delta t$:

$$\int_{0V} {}^{t+\Delta t}{}_0S_{ij} \delta^{t+\Delta t}{}_0\epsilon_{ij} d^0V = {}^{t+\Delta t}\mathcal{R} \quad (12.17)$$

Look at $\delta_0^t \epsilon_{ij}$:

$$\delta_0^t \epsilon_{ij} = \delta \frac{1}{2} ({}^t u_{i,j} + {}_0 u_{j,i} + {}_0 u_{k,i} {}^t u_{k,j}) \quad (12.18a)$$

$$\delta_0^t \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial \delta u_i}{\partial^0 x_j} + \frac{\partial \delta u_j}{\partial^0 x_i} + \frac{\partial \delta u_k}{\partial^0 x_i} \cdot \frac{\partial^t u_k}{\partial^0 x_j} + \frac{\partial^t u_k}{\partial^0 x_i} \cdot \frac{\partial \delta u_k}{\partial^0 x_j} \right) \quad (12.18b)$$

$$\delta_0^t \epsilon_{ij} = \frac{1}{2} (\delta_0 u_{i,j} + \delta_0 u_{j,i} + \delta_0 u_{k,i} {}^t u_{k,j} + {}^t u_{k,i} \delta_0 u_{k,j}) \quad (12.18c)$$

We have

$${}^{t+\Delta t}{}_0\epsilon_{ij} - {}^t{}_0\epsilon_{ij} = {}_0\epsilon_{ij} \quad (12.19)$$

$${}_0\epsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \quad (12.20)$$

where ${}_0e_{ij}$ is the linear incremental strain, ${}_0\eta_{ij}$ is the nonlinear incremental strain, and

$${}_0e_{ij} = \frac{1}{2} \left({}_0u_{i,j} + {}_0u_{j,i} + \underbrace{{}_0^t u_{k,i} {}_0^t u_{k,j} + {}_0^t u_{k,i} {}_0^t u_{k,j}}_{\text{initial displ. effect}} \right) \quad (12.21)$$

$${}_0\eta_{ij} = \frac{1}{2} {}_0^t u_{k,i} {}_0^t u_{k,j} \quad (12.22)$$

where

$${}_0^t u_{k,j} = \frac{\partial u_k}{\partial x_j}, \quad \boxed{u_k = {}^{t+\Delta t}u_k - {}^t u_k} \quad (12.23)$$

Note

$$\delta^{t+\Delta t} \epsilon_{ij} = \delta_0 \epsilon_{ij} \quad (\because \delta_0^t \epsilon_{ij} = 0 \text{ when changing the configuration at } t + \Delta t) \quad (12.24)$$

From (12.17):

$$\begin{aligned} \int_{0V} ({}_0^t S_{ij} + {}_0 S_{ij}) (\delta_0 e_{ij} + \delta_0 \eta_{ij}) d^0V \\ = \int_{0V} ({}_0^t S_{ij} \delta_0 e_{ij} + {}_0 S_{ij} \delta_0 e_{ij} + {}_0^t S_{ij} \delta_0 \eta_{ij} + {}_0 S_{ij} \delta_0 \eta_{ij}) d^0V \end{aligned} \quad (12.25)$$

$$= {}^{t+\Delta t} \mathcal{R} \quad (12.26)$$

Linearization

$$\int_{0V} \left(\underbrace{{}_0^t \mathbf{K}_L \mathbf{U}}_{{}_0^t S_{ij} \delta_0 e_{ij}} + \underbrace{{}_0^t \mathbf{K}_{NL} \mathbf{U}}_{{}_0^t S_{ij} \delta_0 \eta_{ij}} \right) d^0V = {}^{t+\Delta t} \mathcal{R} - \underbrace{\int_{0V} {}_0^t S_{ij} \delta_0 e_{ij} d^0V}_{{}_0^t \mathbf{F}} \quad (12.27)$$

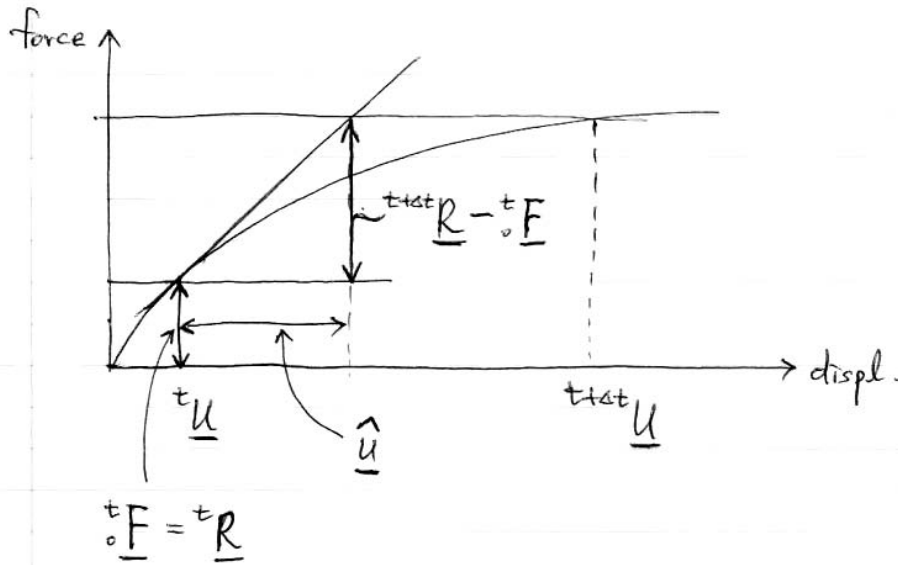
We use,

$${}_0 S_{ij} \simeq {}_0 C_{ijrs} {}_0 e_{rs} \quad (12.28)$$

We arrive at, with the finite element interpolations,

$$({}_0^t \mathbf{K}_L + {}_0^t \mathbf{K}_{NL}) \mathbf{U} = {}^{t+\Delta t} \mathbf{R} - {}_0^t \mathbf{F} \quad (12.29)$$

where \mathbf{U} is the nodal displacement increment.



Left hand side as before but using $(k-1)$ and right hand side is

$$= {}^{t+\Delta t} \underline{R} - \int_{0V} {}^{t+\Delta t} S_{ij} \delta {}^{t+\Delta t} \epsilon_{ij}^{(k-1)} d^0V \quad (12.30)$$

gives

$${}^{t+\Delta t} \underline{R} - {}^{t+\Delta t} \underline{F}^{(k-1)} \quad (12.31)$$

In the full N-R iteration, we use

$$\left({}^{t+\Delta t} \underline{K}_L^{(k-1)} + {}^{t+\Delta t} \underline{K}_{NL}^{(k-1)} \right) \Delta \underline{U}^{(k)} = {}^{t+\Delta t} \underline{R} - {}^{t+\Delta t} \underline{F}^{(k-1)} \quad (12.32)$$