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2.094 Finite Element Analysis of Solids and Fluids  
Spring 2008

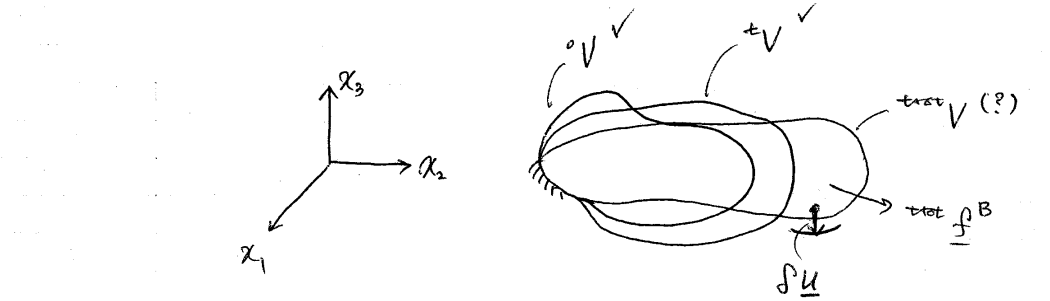
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## Lecture 13 - Total Lagrangian formulation, cont'd

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Example truss element. Recall:

Principle of virtual displacements applied at some time  $t + \Delta t$ :

$$\int_{t+\Delta t V} {}^{t+\Delta t} \tau_{ij} \delta {}_{t+\Delta t} e_{ij} d {}^{t+\Delta t} V = {}^{t+\Delta t} \mathcal{R} \quad (13.1)$$

$$\int_{{}^0 V} {}^{t+\Delta t} S_{ij} \delta {}^{t+\Delta t} {}_0 e_{ij} \delta {}^0 V = {}^{t+\Delta t} \mathcal{R} \quad (13.2)$$

$${}^{t+\Delta t} {}_0 S_{ij} = {}^t S_{ij} + {}_0 S_{ij} \quad (13.3)$$

$${}^{t+\Delta t} {}_0 e_{ij} = {}^t e_{ij} + {}_0 e_{ij} \quad (13.4)$$

$${}_0 e_{ij} = {}_0 e_{ij} + {}_0 \eta_{ij} \quad (13.5)$$

where  ${}^t S_{ij}$  and  ${}^t e_{ij}$  are known, but  ${}_0 S_{ij}$  and  ${}_0 e_{ij}$  are not.

$${}_0 e_{ij} = \frac{1}{2} ({}_0 u_{i,j} + {}_0 u_{j,i} + {}^t u_{k,i} {}_0 u_{k,j} + {}^t u_{k,j} {}_0 u_{k,i}) \quad (13.6)$$

$${}_0 \eta_{ij} = \frac{1}{2} ({}_0 u_{k,i} {}_0 u_{k,j}) \quad (13.7)$$

Substitute into (13.2) and linearize to obtain

$$\int_{{}^0 V} \delta {}_0 e_{ij} {}_0 C_{ijrs} {}_0 e_{rs} d {}^0 V + \int_{{}^0 V} {}^t S_{ij} \delta {}_0 \eta_{ij} d {}^0 V = {}^{t+\Delta t} \mathcal{R} - \int_{{}^0 V} \delta {}_0 e_{ij} {}^t S_{ij} d {}^0 V \quad (13.8)$$

**F.E. discretization gives**

$$({}^t \mathbf{K}_L + {}^t \mathbf{K}_{NL}) \Delta \mathbf{U} = {}^{t+\Delta t} \mathbf{R} - {}^t \mathbf{F} \quad (13.9)$$

$${}^t\mathbf{K}_L = \int_{0V} {}^t\mathbf{B}_L^T {}_0\mathbf{C} {}^t\mathbf{B}_L d^0V \quad (13.10)$$

$${}^t\mathbf{K}_{NL} = \int_{0V} {}^t\mathbf{B}_{NL}^T \underbrace{{}^t\mathbf{S}}_{\text{matrix}} {}^t\mathbf{B}_{NL} d^0V \quad (13.11)$$

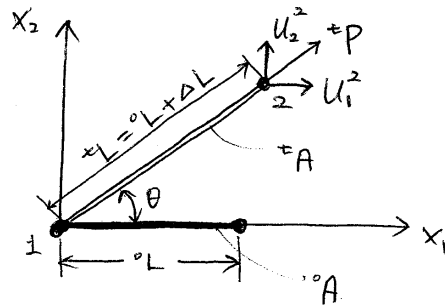
$${}^t\mathbf{F} = \int_{0V} {}^t\mathbf{B}_L^T \underbrace{{}^t\hat{\mathbf{S}}}_{\text{vector}} d^0V \quad (13.12)$$

The iteration (full Newton-Raphson) is

$$\left( {}^{t+\Delta t}{}^t\mathbf{K}_L^{(i-1)} + {}^{t+\Delta t}{}^t\mathbf{K}_{NL}^{(i-1)} \right) \Delta \mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}{}^t\mathbf{F}^{(i-1)} \quad (13.13)$$

$${}^{t+\Delta t}\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{U}^{(i-1)} + \Delta \mathbf{U}^{(i)} \quad (13.14)$$

**Truss element example** (p. 545)



Here we have to only deal with  ${}^tS_{11}$ ,  ${}^0e_{11}$ ,  ${}^0\eta_{11}$

$${}^0e_{11} = \frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial {}^t u_k}{\partial {}^0x_1} \cdot \frac{\partial u_k}{\partial {}^0x_1} \quad (13.15)$$

$${}^0\eta_{11} = \frac{1}{2} \left( \frac{\partial u_k}{\partial {}^0x_1} \cdot \frac{\partial u_k}{\partial {}^0x_1} \right) \quad (13.16)$$

We are after

$${}^0e_{11} = {}^t\mathbf{B}_L \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{pmatrix} = {}^t\mathbf{B}_L \hat{\mathbf{u}} \quad (13.17)$$

$$u_i = \sum_{k=1}^2 h_k u_i^k \quad (13.18)$$

$${}^t u_i = \sum_{k=1}^2 h_k {}^t u_i^k \quad (13.19)$$

$${}^0e_{11} = \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial^t u_1}{\partial^0 x_1} \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial^t u_2}{\partial^0 x_1} \frac{\partial u_2}{\partial^0 x_1} \quad (13.20a)$$

$${}^t u_1^2 = ({}^0L + \Delta L) \cos \theta - {}^0L \quad (13.20b)$$

$${}^t u_2^2 = ({}^0L + \Delta L) \sin \theta \quad (13.20c)$$

$$\begin{aligned} {}^0e_{11} &= \frac{1}{{}^0L} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \hat{\mathbf{u}} \\ &+ \left( \underbrace{\frac{{}^0L + \Delta L}{{}^0L} \cos \theta - 1}_{\frac{\partial^t u_1}{\partial^0 x_1}} \right) \cdot \frac{1}{{}^0L} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \hat{\mathbf{u}} \\ &+ \left( \underbrace{\frac{{}^0L + \Delta L}{{}^0L} \sin \theta}_{\frac{\partial^t u_2}{\partial^0 x_1}} \right) \cdot \frac{1}{{}^0L} \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} \hat{\mathbf{u}} \end{aligned} \quad (13.20d)$$

$$= {}^t\mathbf{B}_L \hat{\mathbf{u}} \quad (13.20e)$$

Hence,

$${}^0e_{11} = \boxed{\frac{{}^0L + \Delta L}{({}^0L)^2} \begin{bmatrix} -\cos \theta & -\sin \theta & \cos \theta & \sin \theta \end{bmatrix}} \hat{\mathbf{u}} \quad (13.20f)$$

where the boxed quantity above equals  ${}^t\mathbf{B}_L$ . In small strain but large rotation analysis we assume  $\Delta L \ll {}^0L$ ,

$${}^0e_{11} = \frac{1}{{}^0L} \begin{bmatrix} -\cos \theta & -\sin \theta & \cos \theta & \sin \theta \end{bmatrix} \hat{\mathbf{u}} \quad (13.20g)$$

$${}^0\eta_{11} = \frac{1}{2} \left( \frac{\partial u_1}{\partial^0 x_1} \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial u_2}{\partial^0 x_1} \frac{\partial u_2}{\partial^0 x_1} \right) \quad (13.21a)$$

$$\delta_0 \eta_{11} = \frac{1}{2} \left( \frac{\partial \delta u_1}{\partial^0 x_1} \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial u_1}{\partial^0 x_1} \frac{\partial \delta u_1}{\partial^0 x_1} + \frac{\partial \delta u_2}{\partial^0 x_1} \frac{\partial u_2}{\partial^0 x_1} + \frac{\partial u_2}{\partial^0 x_1} \frac{\partial \delta u_2}{\partial^0 x_1} \right) \quad (13.21b)$$

$$= \left( \frac{\partial \delta u_1}{\partial^0 x_1} \frac{\partial u_1}{\partial^0 x_1} + \frac{\partial \delta u_2}{\partial^0 x_1} \frac{\partial u_2}{\partial^0 x_1} \right) \quad (13.21c)$$

$${}^tS_{11} \delta_0 \eta_{11} = \begin{bmatrix} \frac{\partial \delta u_1}{\partial^0 x_1} & \frac{\partial \delta u_2}{\partial^0 x_1} \end{bmatrix} \underbrace{\begin{pmatrix} {}^tS_{11} & 0 \\ 0 & {}^tS_{11} \end{pmatrix}}_{{}^t\mathbf{S}} \begin{pmatrix} \frac{\partial u_1}{\partial^0 x_1} \\ \frac{\partial u_2}{\partial^0 x_1} \end{pmatrix} \quad (13.21d)$$

$$\begin{pmatrix} \frac{\partial u_1}{\partial^0 x_1} \\ \frac{\partial u_2}{\partial^0 x_1} \end{pmatrix} = \underbrace{\frac{1}{{}^0L} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\mathbf{B}_{NL}} \hat{\mathbf{u}} \quad (13.21e)$$

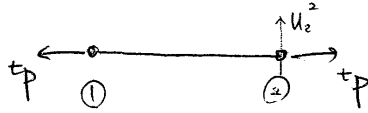
$${}^0\mathbf{C} = E \quad (13.22)$$

$${}^t\hat{\mathbf{S}} = {}^t\mathbf{S}_{11} \quad (13.23)$$

Assume small strains

$${}^t\mathbf{K} = \frac{EA}{{}^0L} \underbrace{\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ \text{sym} & & \cos^2\theta & \sin\theta\cos\theta \\ & & & \sin^2\theta \end{bmatrix}}_{{}^t\mathbf{K}_L} \quad (13.24)$$

$$+ \frac{{}^tP}{{}^0L} \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{{}^t\mathbf{K}_{NL}}$$



When  $\theta = 0$ ,  ${}^t\mathbf{K}_L$  doesn't give stiffness corresponding to  $u_2^2$ , but  ${}^t\mathbf{K}_{NL}$  does.