2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

# Lecture 15 - Field problems

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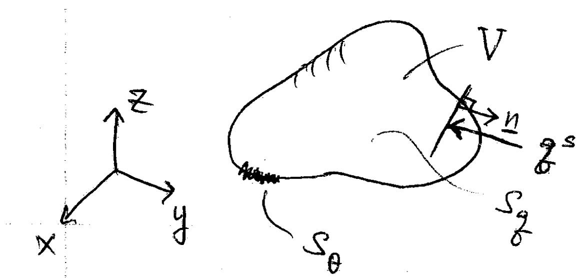
Heat transfer, incompressible/inviscid/irrotational flow, seepage flow, etc.

Reading:
Sec. 7.2-7.3

- Differential formulation
- Variational formulation
- Incremental formulation
- F.E. discretization

## 15.1 Heat transfer




Assume $V$ constant for now:

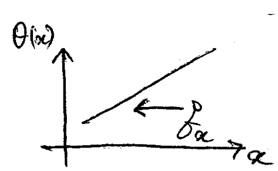
$$S = S_\theta \cup S_q$$

$\theta(x, y, z, t)$ is unkown except $\theta|_{S_\theta} = \theta_{pr}$. In addition, $q^s|_{S_q}$ is also prescribed.

### 15.1.1 Differential formulation

I. Heat flow equilibrium in $V$ and on $S_q$.

II. Constitutive laws $q_x = -k\frac{\partial\theta}{\partial x}$.




$$q_y = -k\frac{\partial\theta}{\partial y} \tag{15.1}$$

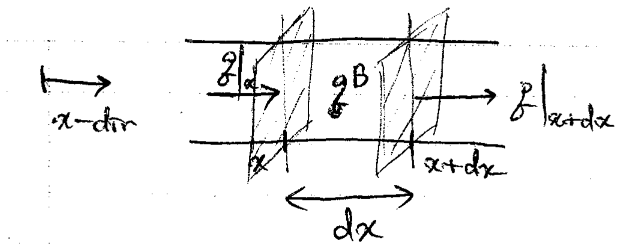
$$q_z = -k\frac{\partial\theta}{\partial z} \tag{15.2}$$

III. Compatibility: temperatures need to be continuous and satisfy the boundary conditions.

Heat flow equilibrium gives

$$\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial \theta}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial \theta}{\partial z}\right)=-q^B \tag{15.3}$$

where $q^B$ is the heat generated per unit volume. Recall 1D case:




unit cross-section

$$dV = dx\cdot(1) \tag{15.4}$$

$$q|_x - q|_{x+dx} + q^B dx = 0 \tag{15.5}$$

$$q|_x - \left(q|_x + \frac{\partial q_x}{\partial x}dx\right) + q^B dx = 0 \tag{15.6}$$

$$-\frac{\partial}{\partial x}\left(-k\frac{\partial \theta}{\partial x}\right)\cancel{dx} + q^B \cancel{dx} = 0 \tag{15.7}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right) = -q^B \tag{15.8}$$

We also need to satisfy

$$k\frac{\partial \theta}{\partial n} = q^S \tag{15.9}$$

on $S_q$.


### 15.1.2 Principle of virtual temperatures

$$\overline{\theta}\left(\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right)+\cdots+q^B\right)=0 \tag{15.10}$$

($\overline{\theta}\big|_{S_\theta}=0$ and $\overline{\theta}$ to be continuous.)

$$\int_V \overline{\theta}\left(\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right)+\cdots+q^B\right)dV=0 \tag{15.11}$$

Transform using divergence theorem (see Ex 4.2, 7.1)

$$\int_V \overline{\boldsymbol{\theta}}'^T \underbrace{\boldsymbol{k}\boldsymbol{\theta}'}_{\text{heat flow}} dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^{S_q} q^S dS_q \tag{15.12}$$

$$\boldsymbol{\theta}' = \begin{pmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \theta}{\partial z} \end{pmatrix} \tag{15.13}$$

$$\boldsymbol{k} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \tag{15.14}$$

**Convection boundary condition**

$$q^S = h\left(\theta^e - \theta^S\right) \tag{15.15}$$

where $\theta^e$ is the given environmental temperature.

**Radiation**

$$q^S = \kappa^* \left[(\theta^r)^4 - \left(\theta^S\right)^4\right] \tag{15.16}$$

$$= \kappa^* \left[(\theta^r)^2 + \left(\theta^S\right)^2\right]\left(\theta^r + \theta^S\right)\left(\theta^r - \theta^S\right) \tag{15.17}$$

$$= \kappa\left(\theta^r - \theta^S\right) \tag{15.18}$$

where $\kappa = \kappa(\theta^S)$ and $\theta^r$ is given temperature of source. At time $t + \Delta t$

$$\int_V \overline{\boldsymbol{\theta}}'^T \, {}^{t+\Delta t}\boldsymbol{k} \, {}^{t+\Delta t}\boldsymbol{\theta}' dV = \int_V \overline{\theta}\, {}^{t+\Delta t}q^B dV + \int_{S_q} \overline{\theta}^S \, {}^{t+\Delta t}q^S dS_q \tag{15.19}$$

$$\text{Let} \quad {}^{t+\Delta t}\theta = {}^t\theta + \theta \tag{15.20}$$

$$\text{or} \quad {}^{t+\Delta t}\theta^{(i)} = {}^{t+\Delta t}\theta^{(i-1)} + \Delta\theta^{(i)} \tag{15.21}$$

$$\text{with} \quad {}^{t+\Delta t}\theta^{(0)} = {}^t\theta \tag{15.22}$$

From (15.19)

$$\begin{aligned} &\int_V \overline{\boldsymbol{\theta}}'^T \, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \Delta\boldsymbol{\theta}'^{(i)} dV \\ &= \int_V \overline{\theta}\, {}^{t+\Delta t}q^B dV - \int_V \overline{\boldsymbol{\theta}}'^T \, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \, {}^{t+\Delta t}\boldsymbol{\theta}'^{(i-1)} dV \\ &\quad + \int_{S_q} \boxed{\overline{\theta}^S \, {}^{t+\Delta t}h^{(i-1)}} \left({}^{t+\Delta t}\theta^e - \left({}^{t+\Delta t}\theta^{S^{(i-1)}} + \boxed{\Delta\theta^{S^{(i)}}}\right)\right) dS_q \end{aligned} \tag{15.23}$$

where the $\Delta\theta^{S^{(i)}}$ term would be moved to the left-hand side.

We considered the convection conditions

$$\int_{S_q} \overline{\theta}^S \, {}^{t+\Delta t}h \left({}^{t+\Delta t}\theta^e - {}^{t+\Delta t}\theta^S\right) dS_q \tag{15.24}$$

The radiation conditions would be included similarly.

**F.E. discretization**

$$
\begin{aligned}
{}^{t+\Delta t}\theta &= \boldsymbol{H}_{1\text{x}4} \cdot {}^{t+\Delta t}\hat{\boldsymbol{\theta}}_{4\text{x}1} \qquad \text{for 4-node 2D planar element} && (15.25)\\
{}^{t+\Delta t}\boldsymbol{\theta}'_{2\text{x}1} &= \boldsymbol{B}_{2\text{x}4} \cdot {}^{t+\Delta t}\hat{\boldsymbol{\theta}}_{4\text{x}1} && (15.26)\\
{}^{t+\Delta t}\theta^{S} &= \boldsymbol{H}^{S} \cdot {}^{t+\Delta t}\hat{\boldsymbol{\theta}} && (15.27)
\end{aligned}
$$

For (15.23)

$$
\int_V \overline{\boldsymbol{\theta}}'^{T}\, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \Delta \boldsymbol{\theta}'^{(i)} dV \overset{\text{gives}}{\Longrightarrow} \left( \int_V \underbrace{\boldsymbol{B}^T}_{4\text{x}2} \underbrace{{}^{t+\Delta t}\boldsymbol{k}^{(i-1)}}_{2\text{x}2} \underbrace{\boldsymbol{B}}_{2\text{x}4} dV \right) \Delta \underbrace{\hat{\boldsymbol{\theta}}^{(i)}}_{4\text{x}1} \qquad (15.28)
$$

$$
\int_V \overline{\theta}\, {}^{t+\Delta t} q^B dV \Rightarrow \int_V \boldsymbol{H}^T\, {}^{t+\Delta t} q^B dV \qquad (15.29)
$$

$$
\int_V \overline{\boldsymbol{\theta}}'^{T}\, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)}\, {}^{t+\Delta t}\boldsymbol{\theta}'^{(i-1)} dV \Rightarrow \left( \int_V \boldsymbol{B}^T\, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \boldsymbol{B} dV \right) \underbrace{{}^{t+\Delta t}\hat{\boldsymbol{\theta}}^{(i-1)}}_{\text{known}} \qquad (15.30)
$$

$$
\begin{aligned}
&\int_{S_q} \overline{\theta}^{S^T}\, {}^{t+\Delta t}h^{(i-1)} \left( {}^{t+\Delta t}\theta^e - \left( {}^{t+\Delta t}\theta^{S^{(i-1)}} + \Delta \theta^{S^{(i)}} \right) \right) dS_q \quad \Longrightarrow \\
&\quad \int_{S_q} \underbrace{\boldsymbol{H}^{S^T}}_{4\text{x}1}\, {}^{t+\Delta t}h^{(i-1)} \underbrace{\boldsymbol{H}^S}_{1\text{x}4} \left( \underbrace{{}^{t+\Delta t}\hat{\boldsymbol{\theta}}^e}_{4\text{x}1} - \left( \underbrace{{}^{t+\Delta t}\hat{\boldsymbol{\theta}}^{(i-1)}}_{4\text{x}1} + \Delta \underbrace{\hat{\boldsymbol{\theta}}^{(i)}}_{4\text{x}1} \right) \right) dS_q
\end{aligned} \qquad (15.31)
$$

## 15.2 Inviscid, incompressible, irrotational flow

2D case: $v_x$, $v_y$ are velocities in $x$ and $y$ directions.

Reading: Sec. 7.3.2

$$
\begin{aligned}
&\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 && (15.32)\\
\text{or} \quad &\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{(incompressible)} && (15.33)\\
&\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0 \qquad \text{(irrotational)} && (15.34)
\end{aligned}
$$

Use the potential $\phi(x, y)$,

$$
v_x = \frac{\partial \phi}{\partial x} \qquad v_y = \frac{\partial \phi}{\partial y} \qquad (15.35)
$$

$$
\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{in } V \qquad (15.36)
$$

(Same as the heat transfer equation with $k = 1$, $q^B = 0$)