2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids	Fall '08
Lecture 16 - F.E. analysis of Navier-Stokes fluids	
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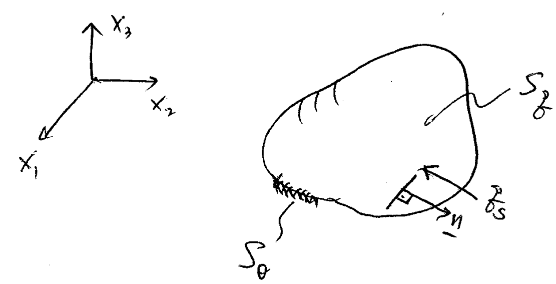
Incompressible flow with heat transfer

Reading: Sec. 7.1-7.4, Table 7.3

We recall heat transfer for a solid:



Governing differential equations

$$(k\theta_{,i})_{,i} + q^B = 0 \qquad \text{in } V \tag{16.1}$$

$$\theta\Big|_{S_\theta} \text{ is prescribed, } k\frac{\partial \theta}{\partial n}\Big|_{S_q} = q^S\Big|_{S_q} \tag{16.2}$$

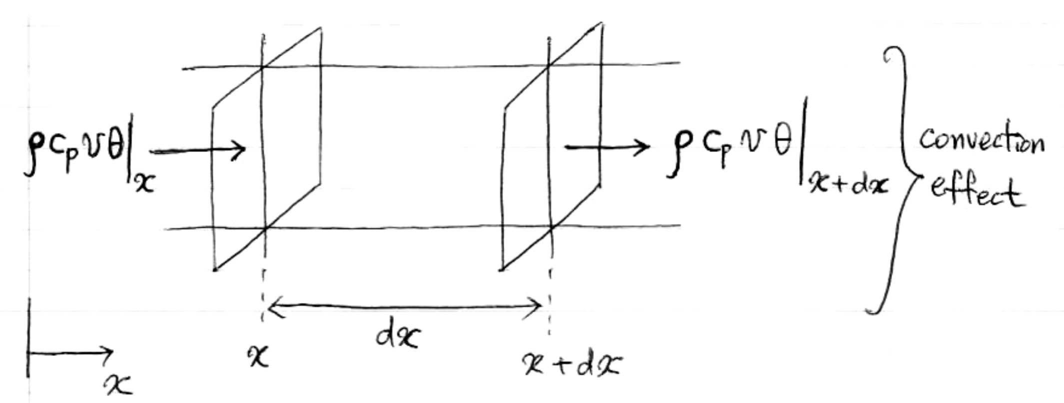
$$\mathrm{S}_\theta \cup \mathrm{S}_q = \mathrm{S} \qquad \mathrm{S}_\theta \cap \mathrm{S}_q = \varnothing \tag{16.3}$$

Principle of virtual temperatures

$$\int_V \overline{\theta}_{,i} k\theta_{,i} dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^S q^S dS_q \tag{16.4}$$

for arbitrary continuous $\overline{\theta}(x_1, x_2, x_3)$ zero on S_θ

For a fluid, we use the Eulerian formulation.



$$\rho c_p v\ \theta|_x - \left\{ \rho c_p v\ \theta|_x + \frac{\partial}{\partial x}(\rho c_p v \theta) dx \right\} +\ \text{conduction}\ +\ \text{etc} \tag{16.5}$$

In general 3D, we have an additional term for the left hand side of (16.1):

$$-\boldsymbol{\nabla}\cdot(\rho c_p \boldsymbol{v}\theta) = -\rho c_p \boldsymbol{\nabla}\cdot(\boldsymbol{v}\theta) = -\rho c_p(\cancel{\boldsymbol{\nabla}\cdot\boldsymbol{v}})\theta - \underbrace{\rho c_p\,(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\theta}_{\text{term (A)}} \tag{16.6}$$

where $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ in the incompressible case.

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = v_{i,i} = \mathrm{div}(\boldsymbol{v}) = 0 \tag{16.7}$$

So (16.1) becomes

$$\left(k\theta_{,i}\right)_{,i} + q^B = \rho c_p \theta_{,i} v_i \Rightarrow \left(k\theta_{,i}\right)_{,i} + \left(q^B - \rho c_p \theta_{,i} v_i\right) = 0 \tag{16.8}$$

Principle of virtual temperatures is now (use (16.4))

$$\int_V \overline{\theta}_{,i} k \theta_{,i} dV + \int_V \overline{\theta}\left(\rho c_p \theta_{,i} v_i\right) dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^S q^S dS_q \tag{16.9}$$

Navier-Stokes equations

- *Differential form*

$$\tau_{ij,j} + f_i^B = \rho v_{i,j} v_j \tag{16.10}$$

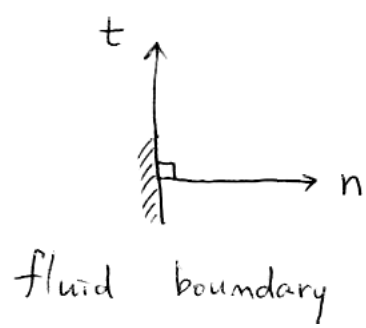
  with $\rho v_{i,j} v_j$ like term (A) in (16.6) $= \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ in V.

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{16.11}$$

- *Boundary conditions* (need be modified for various flow conditions)

$$\tau_{ij} n_j = f_i^{S_f} \text{ on } S_f \tag{16.12}$$

  Mostly used as $f_n = \tau_{nn}$ = prescribed, f_t = unknown with possibly $\frac{\partial v_n}{\partial n} = \frac{\partial v_t}{\partial n} = 0$ (outflow or inflow conditions).



  And v_i prescribed on S_v, and $\mathrm{S}_v \cup \mathrm{S}_f = \mathrm{S}$ and $\mathrm{S}_v \cap \mathrm{S}_f = \varnothing$.

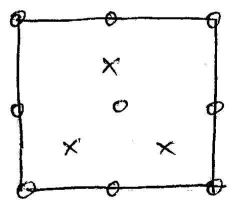
- *Variational form*

$$\int_V \overline{v}_i \rho v_{i,j} v_j dV + \int_V \overline{e}_{ij} \tau_{ij} dV = \int_V \overline{v}_i f_i^B dV + \int_{S_f} \overline{v}_i^{S_f} f_i^{S_f} dS_f \tag{16.13}$$

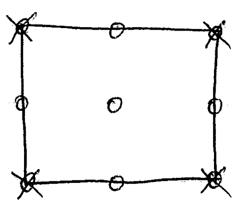
$$\int_V \overline{p} \boldsymbol{\nabla} \cdot \boldsymbol{v} dV = 0 \tag{16.14}$$

- *F.E. solution*

  We interpolate (x_1, x_2, x_3), v_i, $\overline{v}_i$, θ, $\overline{\theta}$, p, $\overline{p}$. Good elements are



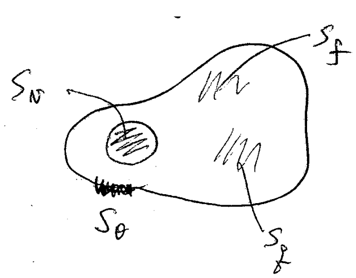
×: linear pressure
○: biquadratic velocities
(Q_2, P_1), 9/3 element



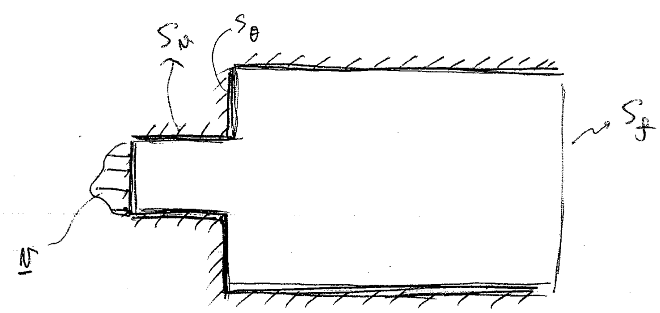
9/4c element

  Both satisfy the inf-sup condition.

So in general,



Example:

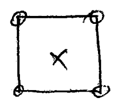


For S_f e.g.

$$\tau_{nn} = 0, \qquad \frac{\partial v_t}{\partial n} = 0; \tag{16.15}$$

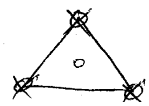
and $\frac{\partial v_n}{\partial t}$ is solved for. Actually, we frequently just set $p = 0$.

Frequently used is the 4-node element with constant pressure



It does not strictly satisfy the inf-sup condition. Or use

Reading: Sec. 7.4

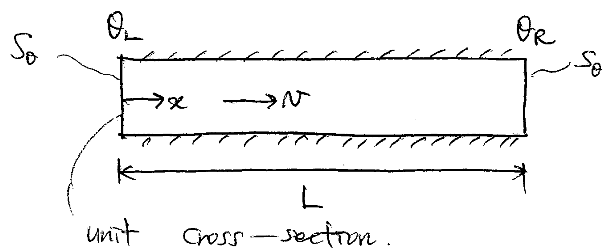


3-node element with a bubble node.
Satisfies inf-sup condition

1D case of heat transfer with fluid flow, $v =$ constant

Reading: Sec. 7.4.3

$$\mathrm{Re} = \frac{vL}{\nu} \qquad \mathrm{Pe} = \frac{vL}{\alpha} \qquad \alpha = \frac{k}{\rho c_p} \tag{16.16}$$



$S_f = \phi$.

- *Differential equations*

$$k\theta'' = \rho c_p \theta' v \tag{16.17}$$

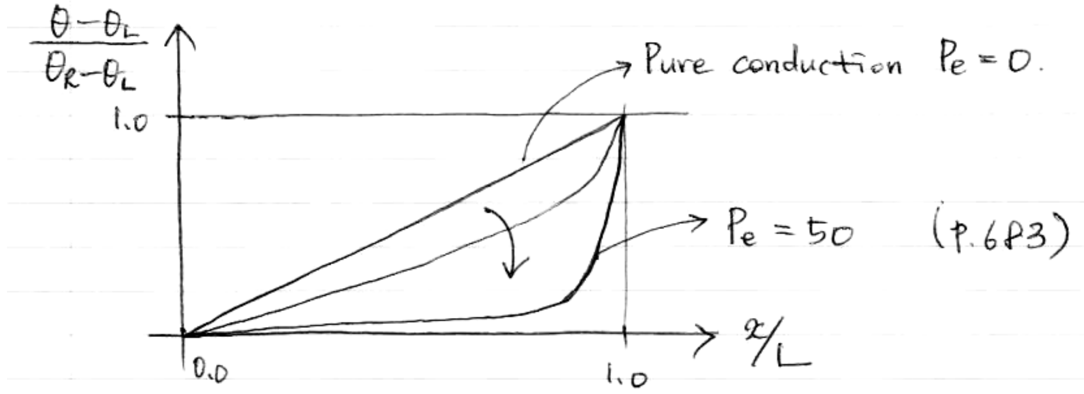
$$\theta|_{x=0} = \theta_L \qquad \theta|_{x=L} = \theta_R \tag{16.18}$$

In non-dimensional form

Reading: p. 683

$$\boxed{\frac{1}{\mathrm{Pe}}\theta'' = \theta'} \qquad \text{(now } \theta'' \text{ and } \theta' \text{ are non-dimensional)} \tag{16.19}$$

$$\Rightarrow \frac{\theta - \theta_L}{\theta_R - \theta_L} = \frac{\exp\left(\frac{\mathrm{Pe}}{L}x\right) - 1}{\exp(\mathrm{Pe}) - 1} \tag{16.20}$$

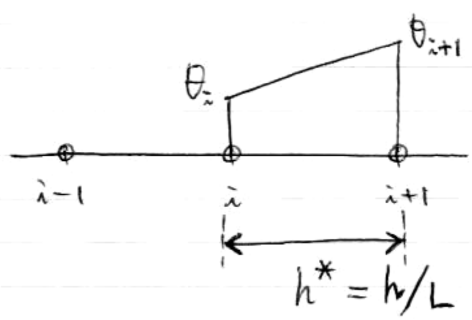


- *F.E. discretization*

$$\theta'' = \mathrm{Pe}\theta' \tag{16.21}$$

$$\int_0^1 \overline{\theta}'\theta' dx + \mathrm{Pe}\int_0^1 \overline{\theta}\theta' dx = 0 + \{\text{ effect of boundary conditions } = 0 \text{ here}\} \tag{16.22}$$

Using 2-node elements gives



$$\frac{1}{(h^*)^2}\left(\theta_{i+1} - 2\theta_i + \theta_{i-1}\right) = \frac{\mathrm{Pe}}{2h^*}\left(\theta_{i+1} - \theta_{i-1}\right) \tag{16.23}$$

$$\mathrm{Pe} = \frac{vL}{\alpha} \tag{16.24}$$

Define

$$\mathrm{Pe}^e = \mathrm{Pe}\cdot\frac{h}{L} = \frac{vh}{\alpha} \tag{16.25}$$

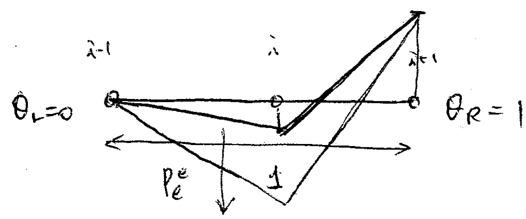
$$\left(-1 - \frac{\mathrm{Pe}^e}{2}\right)\theta_{i-1} + 2\theta_i + \left(\frac{\mathrm{Pe}^e}{2} - 1\right)\theta_{i+1} = 0 \tag{16.26}$$

what is happening when Pe^e is large? Assume two 2-node elements only.

$$\theta_{i-1} = 0 \tag{16.27}$$

$$\theta_{i+1} = 1 \tag{16.28}$$

$$\theta_i = \frac{1}{2}\left(1 - \frac{\mathrm{Pe}^e}{2}\right) \tag{16.29}$$



$$\theta_i = \frac{1}{2}\left(1 - \frac{\mathrm{Pe}^e}{2}\right) \tag{16.30}$$

For $\mathrm{Pe}^e > 2$, we have *negative* θ_i (unreasonable).