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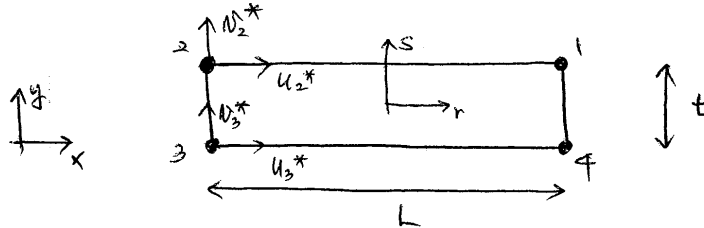
2.094 Finite Element Analysis of Solids and Fluids  
Spring 2008

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Lecture 19 - Slender structures

Beam analysis,  $\frac{t}{L} \ll 1$  (e.g.  $\frac{t}{L} = \frac{1}{100}, \frac{1}{1000}, \dots$ )

Reading:  
Sec. 5.4,  
6.5

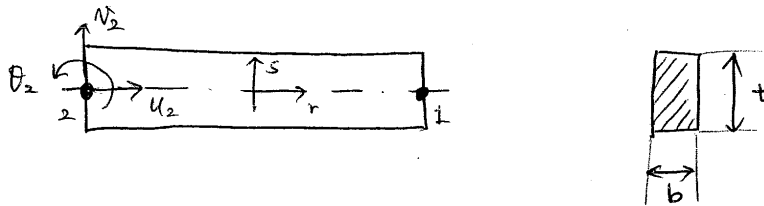


(plane stress)

$$J = \begin{bmatrix} \frac{L}{2} & 0 \\ 0 & \frac{t}{2} \end{bmatrix} \tag{19.1}$$

$$h_2 = \frac{1}{4}(1-r)(1+s) \tag{19.2}$$

$$h_3 = \frac{1}{4}(1-r)(1-s) \tag{19.3}$$



Beam theory assumptions (Timoshenko beam theory):

$$v_2^* = v_3^* = v_2 \tag{19.4}$$

$$u_3^* = u_2 + \frac{t}{2}\theta_2 \tag{19.5}$$

$$u_2^* = u_2 - \frac{t}{2}\theta_2 \tag{19.6}$$

$$B^* = \begin{bmatrix} u_2^* & v_2^* & u_3^* & v_3^* \\ -\frac{1}{4}(1+s)\frac{2}{L} & 0 & -\frac{1}{4}(1-s)\frac{2}{L} & 0 \\ 0 & \frac{1}{4}(1-r)\frac{2}{t} & 0 & -\frac{1}{4}(1-r)\frac{2}{t} \\ \frac{1}{4}(1-r)\frac{2}{t} & -\frac{1}{4}(1+s)\frac{2}{L} & -\frac{1}{4}(1-r)\frac{2}{t} & -\frac{1}{4}(1-s)\frac{2}{L} \end{bmatrix} \text{ etc} \tag{19.7}$$

$$B_{\text{beam}} = \begin{bmatrix} u_2 & v_2 & \theta_2 \\ -\frac{1}{L} & 0 & \frac{t}{2L}s \\ \dots & \emptyset & \emptyset & \emptyset \\ 0 & -\frac{1}{L} & -\frac{1}{2}(1-r) \end{bmatrix} \sim \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \quad (19.8)$$

$$v(r) = \frac{1}{2}(1-r)v_2 \quad (19.9)$$

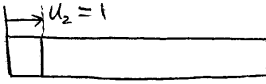
$$u(r) = \frac{1}{2}(1-r)u_2 - \frac{st}{4}(1-r)\theta_2 \quad (19.10)$$

at  $r = -1$ ,

$$v(-1) = v_2 \quad (19.11)$$

$$u(-1) = -\frac{st}{2}\theta_2 + u_2 \quad (19.12)$$

Kinematics is



$$u(r) = \frac{1}{2}(1-r)u_2 \quad (19.13)$$

results into  $\epsilon_{xx}$

$$\rightarrow \epsilon_{xx} = \frac{\partial u}{\partial r} \cdot \frac{2}{L} = -\frac{1}{L} \quad (19.14)$$

$$u(r, s) = -\frac{st}{4}(1-r)\theta_2 \quad (19.15)$$

results into  $\epsilon_{xx}, \gamma_{xy}$

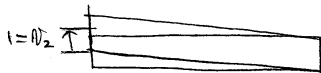
$$\rightarrow \epsilon_{xx} = \frac{st}{2L} \quad (19.16)$$

$$\gamma_{xy} = \frac{\partial u}{\partial s} \cdot \frac{2}{t} = -\frac{1}{2}(1-r) \quad (19.17)$$

$$v(r) = \frac{1}{2}(1-r)v_2 \quad (19.18)$$

results into  $\gamma_{xy}$

$$\rightarrow \gamma_{xy} = -\frac{1}{L} \quad (19.19)$$

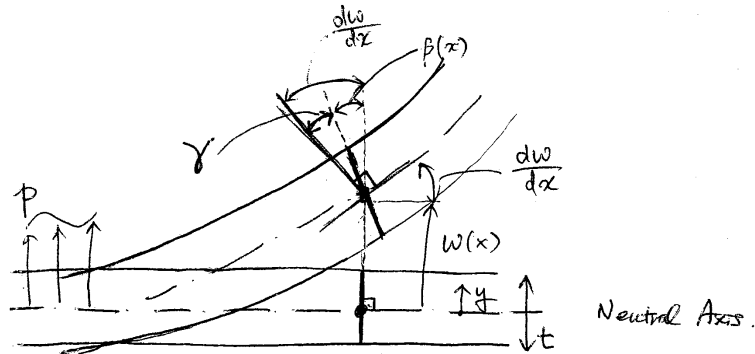


For a pure bending moment, we want

$$-\frac{1}{L}v_2 - \frac{1}{2}(1-r)\theta_2 = 0 \quad (19.20)$$

for all  $r!$   $\Rightarrow$  Impossible (except for  $v_2 = \theta_2 = 0$ )  $\Rightarrow$  So, the element has a spurious shear strain!

**Beam kinematics** (Timoshenko, Reissner-Mindlin)



$$\gamma = \frac{dw}{dx} - \beta \quad (19.21)$$

$$\left( I = \frac{1}{12}bt^3 \right) \quad (19.22)$$

**Principle of virtual work**

$$EI \int_0^L \frac{d\bar{\beta}}{dx} \frac{d\beta}{dx} dx + A_s G \int_0^L \left( \frac{d\bar{w}}{dx} - \bar{\beta} \right) \left( \frac{dw}{dx} - \beta \right) dx = \int_0^L p \bar{w} dx \quad (19.23)$$

$$A_s = kA = kbt \quad (19.24)$$

To calculate  $k$

Reading:  
p. 400

$$\int_A \frac{1}{2G} (\tau_a)^2 dA = \int_{A_s} \frac{1}{2G} \left( \frac{V}{A_s} \right)^2 dA_s \quad (19.25)$$

where  $\tau_a$  is the actual shear stress:

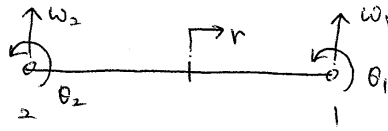
$$\tau_a = \frac{3}{2} \cdot \frac{V}{A} \left[ \frac{\left(\frac{t}{2}\right)^2 - y^2}{\left(\frac{t}{2}\right)^2} \right] \quad (19.26)$$

and  $V$  is the shear force.

Reading:  
Ex. 5.23

$$\Rightarrow k = \frac{5}{6} \quad (19.27)$$

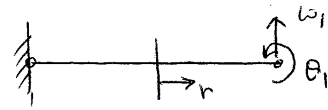
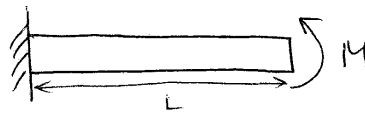
Now interpolate



$$w(r) = h_1 w_1 + h_2 w_2 \quad (19.28)$$

$$\beta(r) = h_1 \theta_1 + h_2 \theta_2 \quad (19.29)$$

Revisit the simple case:



$$w = \frac{1+r}{2} w_1 \quad (19.30)$$

$$\beta = \frac{1+r}{2} \theta_1 \quad (19.31)$$

Shearing strain

$$\gamma = \frac{w_1}{L} - \frac{1+r}{2} \theta_1 \quad (19.32)$$

Shear strain is not zero all along the beam. But, at  $r = 0$ , we can have the shear strain = 0.

$$\frac{w_1}{L} - \frac{\theta_1}{2} \text{ can be zero} \quad (19.33)$$

Namely,

$$\frac{w_1}{L} - \frac{\theta_1}{2} = 0 \quad \text{for} \quad \theta_1 = \frac{2}{L} w_1 \quad (19.34)$$