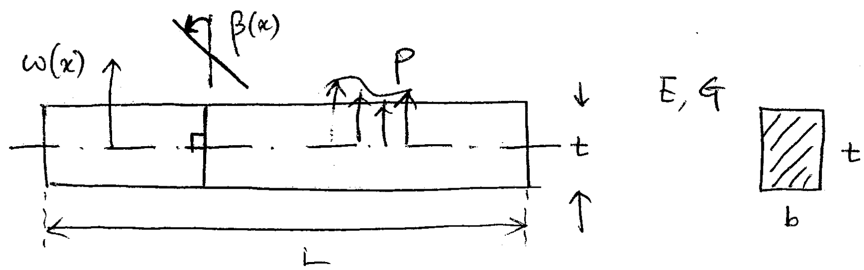
2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

Lecture 20 - Beams, plates, and shells

Prof. K.J. Bathe MIT OpenCourseWare

Timoshenko beam theory

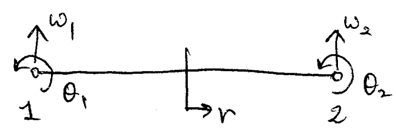


The fiber moves up and rotates and its length does not change.

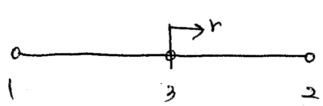
Principle of virtual displacement (Linear Analysis)

$$EI \int_0^L \left(\overline{\beta}'\right)^T \beta' dx + (Ak)G \int_0^L \left(\frac{d\overline{w}}{dx} - \overline{\beta}\right)^T \left(\frac{dw}{dx} - \beta\right) dx = \int_0^L \overline{w}^T p dx \tag{20.1}$$

Two-node element:



Three-node element:



For a q-node element,

$$\hat{\boldsymbol{u}} = \left[\begin{array}{cccccc} w_1 & \cdots & w_q & \theta_1 & \cdots & \theta_q \end{array}\right]^T \tag{20.2}$$

$$w = \boldsymbol{H}_w \hat{\boldsymbol{u}} \tag{20.3}$$

$$\beta = \boldsymbol{H}_\beta \hat{\boldsymbol{u}} \tag{20.4}$$

$$\boldsymbol{H}_w = \left[\begin{array}{cccccc} h_1 & \cdots & h_q & 0 & \cdots & 0 \end{array}\right] \tag{20.5}$$

$$\boldsymbol{H}_\beta = \left[\begin{array}{cccccc} 0 & \cdots & 0 & h_1 & \cdots & h_q \end{array}\right] \tag{20.6}$$

$$\boldsymbol{J} = \frac{dx}{dr} \tag{20.7}$$

$$\frac{dw}{dx} = \underbrace{\boldsymbol{J}^{-1}\boldsymbol{H}_{w,r}}_{\boldsymbol{B}_w}\hat{\boldsymbol{u}} \tag{20.8}$$

$$\frac{d\beta}{dx} = \underbrace{\boldsymbol{J}^{-1}\boldsymbol{H}_{\beta,r}}_{\boldsymbol{B}_\beta}\hat{\boldsymbol{u}} \tag{20.9}$$

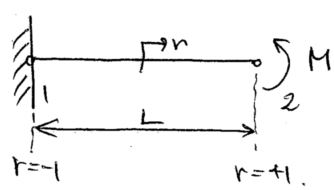
Hence we obtain

$$\left[EI\int_{-1}^{1}\boldsymbol{B}_\beta^T\boldsymbol{B}_\beta\det(\boldsymbol{J})dr + (Ak)G\int_{-1}^{1}(\boldsymbol{B}_w-\boldsymbol{H}_\beta)^T(\boldsymbol{B}_w-\boldsymbol{H}_\beta)\det(\boldsymbol{J})dr\right]\hat{\boldsymbol{u}} = \int_{-1}^{1}\boldsymbol{H}_w^T p\det(\boldsymbol{J})dr \tag{20.10}$$

$$\boxed{\boldsymbol{K}\hat{\boldsymbol{u}} = \boldsymbol{R}} \tag{20.11}$$

$\boldsymbol{K}$ is a result of the term inside the bracket in (20.10) and $\boldsymbol{R}$ is a result of the right hand side.

For the 2-node element,



$$w_1 = \theta_1 = 0 \tag{20.12}$$

$$w_2,\,\theta_2 = ? \tag{20.13}$$

$$\gamma = \frac{w_2}{L} - \frac{1+r}{2}\theta_2 \tag{20.14}$$

We cannot make γ equal to zero for every r (page 404, textbook). Because of this, we need to use about 200 elements to get an error of 10%. (Not good!)

Recall almost or fully incompressible analysis: Principle of virtual displacements:

$$\int_V \overline{\boldsymbol{\epsilon}}'^T\boldsymbol{C}'\boldsymbol{\epsilon}'dV + \int_V \overline{\epsilon}_v\,(\kappa\epsilon_v)dV = \mathcal{R} \tag{20.15}$$

u/p formulation

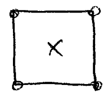
$$\int_V \overline{\boldsymbol{\epsilon}}'^T\boldsymbol{C}'\boldsymbol{\epsilon}'dV - \int_V \overline{\epsilon}_v p dV = \mathcal{R} \tag{20.16}$$

$$\int_V \overline{p}\left(\frac{p}{\kappa}+\epsilon_v\right)dV = 0 \tag{20.17}$$

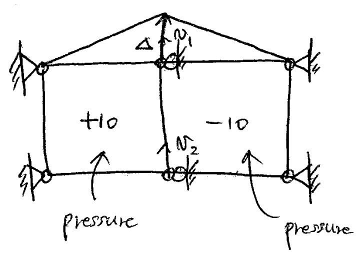
But now we needed to select wisely the interpolations of u and p. We needed to satisfy the inf-sup condition

$$\underbrace{\inf}_{q_h\in \mathrm{Q}_h}\ \underbrace{\sup}_{\boldsymbol{v}_h\in \mathrm{V}_h}\frac{\int_{\mathrm{Vol}} q_h\boldsymbol{\nabla}\cdot\boldsymbol{v}_h d\mathrm{Vol}}{\|q_h\|\|v_h\|} \geq \beta > 0 \tag{20.18}$$

4/1 element:



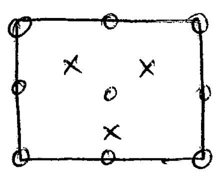
We can show mathematically that this element does not satisfy inf-sup condition. But, we can also show it by giving an example of this element which violates the inf-sup condition.



$v_1 = \triangle$, $\quad v_2 = 0 \Rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v}_h$ for both elements is positive and the same. Now, if I choose pressures as above

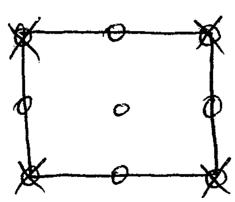
$$\int_{\text{Vol}} q_h \nabla \boldsymbol{v}_h d\text{Vol} = 0, \qquad \text{hence (20.18) is not satisfied!} \tag{20.19}$$

9/3 element



satisfies inf-sup

9/4-c



satisfies inf-sup

Getting back to beams

$$EI \int_0^L \overline{\beta}' \beta dx + (AkG) \int_0^L \left(\frac{d\overline{w}}{dx} - \overline{\beta} \right) \gamma^{AS} dx = \mathcal{R} \tag{20.20}$$

$$\int_0^L \overline{\gamma}^{AS} \left(\gamma - \gamma^{AS} \right) dx = 0 \tag{20.21}$$

where

$$\gamma = \frac{dw}{dx} - \beta, \qquad \text{from displacement interpolation} \tag{20.22}$$

$$\gamma^{AS} = \text{ Assumed shear strain interpolation} \tag{20.23}$$

2-node element, constant shear assumption. From (20.21),

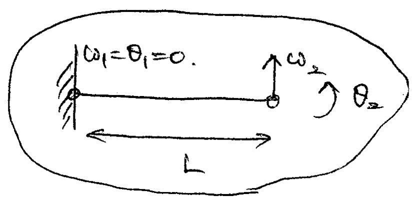
Reading: Sec. 4.5.7

$$\int_0^L \left(\frac{dw}{dx} - \beta\right) \cancel{\overline{\gamma}^{AS}}\, dx = \int_0^L \gamma^{AS} \cancel{\overline{\gamma}^{AS}}\, dx \tag{20.24}$$

$$\Rightarrow -\int_{-1}^{+1} \left(\frac{1+r}{2}\theta_2\right) \cdot \frac{L}{2} dr + w_2 = \gamma^{AS} \cdot L \tag{20.25}$$

$$\Rightarrow \gamma^{AS} = \frac{w_2 - \frac{L}{2}\theta_2}{L} \tag{20.26}$$

γ^{AS} (shear strain) is equal to the displacement-based shear strain at the middle of the beam.



Use γ^{AS} in (20.20) to obtain a powerful element. For "our problem",

$$\gamma^{AS} = 0 \quad \text{hence} \quad w_2 = \frac{L}{2}\theta_2 \tag{20.27}$$

$$\Rightarrow EI \int_0^L \overline{\beta}' \beta' dx = M\, \overline{\beta}\big|_{x=L} \tag{20.28}$$

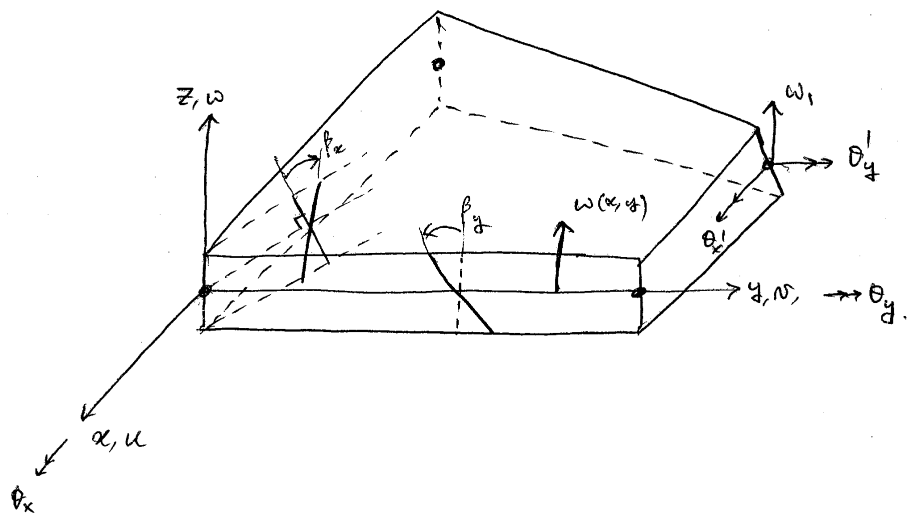
$$\Rightarrow EI \left(\left(\frac{1}{L}\right)^2 \cdot L \right) \theta_2 = M \tag{20.29}$$

$$\Rightarrow \boxed{\theta_2 = \frac{ML}{EI}, \qquad w_2 = \frac{ML^2}{2EI}} \tag{20.30}$$

(exact solutions)

Plates

Reading: Fig. 5.25, p. 421



$$\begin{cases} w = w(x,y) \quad \text{is the transverse displacement of the mid-surface} \\ v = -z\beta_y(x,y) \\ u = -z\beta_x(x,y) \end{cases} \tag{20.31}$$

For any particle in the plate with coordinates (x, y, z), the expressions in (20.31) hold!

We use

$$w = \sum_{i=1}^{q} h_i w_i \tag{20.32}$$

$$\beta_x = -\sum_{i=1}^{q} h_i \theta_y^i \tag{20.33}$$

$$\beta_y = +\sum_{i=1}^{q} h_i \theta_x^i \tag{20.34}$$

where q equals the number of nodes. Then the element locks in the same way as the displacement-based beam element.