2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 3 - Finite element formulation for solids and structures

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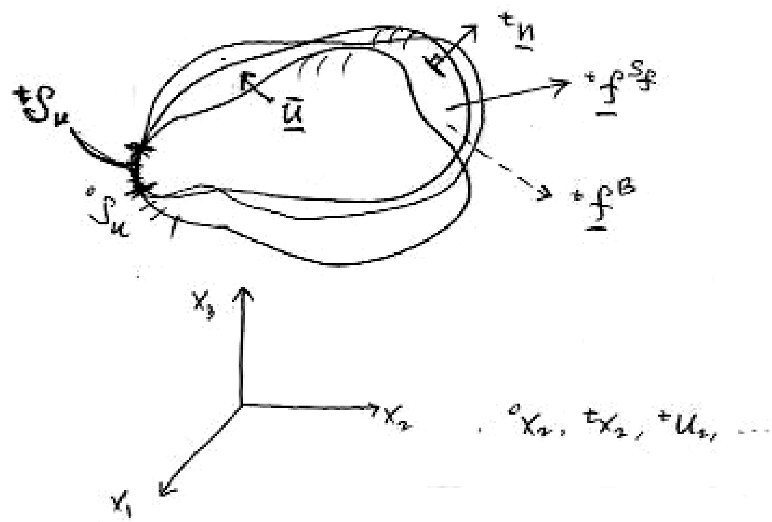


Reading:
Sec. 6.1-6.2

We need to satisfy at time t:

- *Equilibrium*

$$\frac{\partial\, {}^t\tau_{ij}}{\partial\, {}^tx_j} + {}^tf_i^B = 0 \quad (i = 1, 2, 3) \text{ in } {}^tV \tag{3.1}$$

$${}^t\tau_{ij}\, {}^tn_j = {}^tf_i^{S_f} \quad (i = 1, 2, 3) \text{ on } {}^tS_f \tag{3.2}$$

- *Compatibility*
- *Stress-strain law(s)*

Principle of virtual displacements

$$\int_{{}^tV} {}^t\tau_{ij}\, {}_t\overline{e}_{ij}\, d\,{}^tV = \int_{{}^tV} \overline{u}_i\, {}^tf_i^B\, d\,{}^tV + \int_{{}^tS_f} \overline{u}_i|_{{}^tS_f}\, {}^tf_i^{S_f}\, d\,{}^tS_f \tag{3.3}$$

$${}_t\overline{e}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial\, {}^tx_j} + \frac{\partial \overline{u}_j}{\partial\, {}^tx_i}\right) \tag{3.4}$$

- If (3.3) holds for any continuous virtual displacement (zero on ${}^tS_u$), then (3.1) and (3.2) hold and vice versa.
- Refer to Ex. 4.2 in the textbook.

Major steps

I. Take (3.1) and weigh with $\overline{u}_i$:

$$\left({}^{t}\tau_{ij,j} + {}^{t}f_i^B \right) \overline{u}_i = 0. \tag{3.5a}$$

II. Integrate (3.5a) over volume ${}^{t}V$:

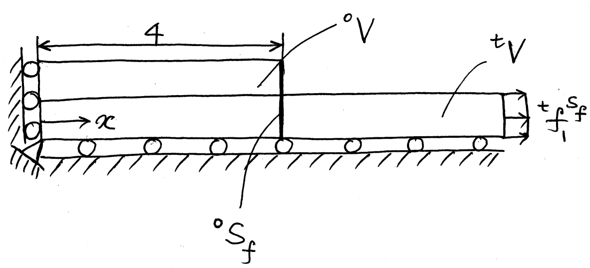
$$\int_{{}^{t}V} \left({}^{t}\tau_{ij,j} + {}^{t}f_i^B \right) \overline{u}_i \, d\,{}^{t}V = 0 \tag{3.5b}$$

III. Use divergence theorem. Obtain a boundary term of stresses times virtual displacements on ${}^{t}S = {}^{t}S_u \cup {}^{t}S_f$.

IV. But, on ${}^{t}S_u$ the $\overline{u}_i = 0$ and on ${}^{t}S_f$ we have (3.2) to satisfy.

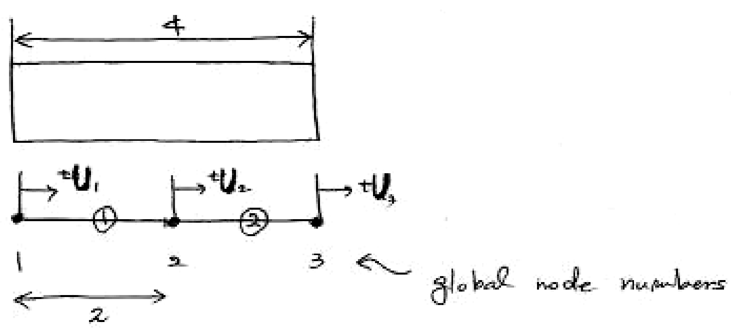
Result: (3.3).

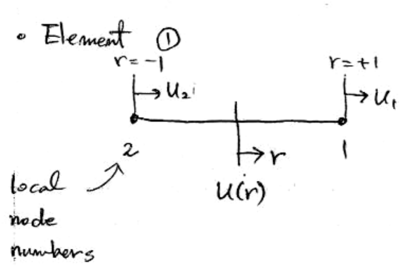
Example



$$\int_{{}^{t}V} {}^{t}\tau_{11}\, {}_{t}\overline{e}_{11} \, d^{t}V = \int_{{}^{t}S_f} \overline{u}_i \, {}^{t}f_1^{S_f} \, d^{t}S_f \tag{3.6}$$

One element solution:





$$u(r) = \frac{1}{2}(1+r)\,u_1 + \frac{1}{2}(1-r)\,u_2 \tag{3.7}$$

$$^{t}u(r) = \frac{1}{2}(1+r)\ {}^{t}u_1 + \frac{1}{2}(1-r)\ {}^{t}u_2 \tag{3.8}$$

$$\overline{u}(r) = \frac{1}{2}(1+r)\,\overline{u}_1 + \frac{1}{2}(1-r)\,\overline{u}_2 \tag{3.9}$$

Suppose we know ${}^{t}\tau_{11}$, ${}^{t}V$, ${}^{t}S_f$, ${}^{t}u$ ... use (3.6).

For element 1,

$${}_{t}\overline{e}_{11} = \frac{\partial \overline{u}}{\partial\, {}^{t}x} = \boldsymbol{B}^{(1)} \begin{pmatrix} \overline{u}_1 \\ \overline{u}_2 \end{pmatrix} \tag{3.10}$$

$$\int_{{}^{t}V} {}_{t}\overline{e}_{11}^{T}\, {}_{t}\tau_{11} d^{t}V \xrightarrow{\text{for el. (1)}} [\overline{u}_1 \quad \overline{u}_2] \underbrace{\int_{{}^{t}V} \boldsymbol{B}^{(1)^T}\ {}^{t}\tau_{11}\ d\,{}^{t}V}_{=\,{}^{t}\boldsymbol{F}^{(1)}} \tag{3.11}$$

$$\xrightarrow{\text{for el. (1)}} [\overline{u}_1 \quad \overline{u}_2]\ {}^{t}\boldsymbol{F}^{(1)} \tag{3.12}$$

$$= \begin{bmatrix} \underbrace{\overline{U}_1}_{\overline{u}_2} & \underbrace{\overline{U}_2}_{\overline{u}_1} & \overline{U}_3 \end{bmatrix} \begin{bmatrix} {}^{t}\hat{\boldsymbol{F}}^{(1)} \\ 0 \end{bmatrix} \tag{3.13}$$

where

$${}^{t}\hat{F}_1^{(1)} = {}^{t}F_2^{(1)} \tag{3.14}$$

$${}^{t}\hat{F}_2^{(1)} = {}^{t}F_1^{(1)} \tag{3.15}$$

For element 2, similarly,

$$= \begin{bmatrix} \overline{U}_1 & \underbrace{\overline{U}_2}_{\overline{u}_2} & \underbrace{\overline{U}_3}_{\overline{u}_1} \end{bmatrix} \begin{bmatrix} 0 \\ {}^{t}\hat{\boldsymbol{F}}^{(2)} \end{bmatrix} \tag{3.16}$$

R.H.S.

$$\underbrace{\begin{bmatrix} \overline{U}_1 & \overline{U}_2 & \overline{U}_3 \end{bmatrix}}_{\overline{\boldsymbol{U}}^T} \begin{bmatrix} \text{(unknown reaction at left)} \\ 0 \\ {}^{t}S_f \cdot {}^{t}f_1^{S_f} \end{bmatrix} \tag{3.17}$$

Now apply,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{3.18}$$

then,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{3.19}$$

then,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{3.20}$$

This gives,

$$\begin{bmatrix} {}^t\hat{\boldsymbol{F}}^{(1)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ {}^t\hat{\boldsymbol{F}}^{(2)} \end{bmatrix} = \begin{bmatrix} \text{unknown reaction} \\ 0 \\ {}^t f_1^{\,{}^tS_f} \cdot {}^tS_f \end{bmatrix} \tag{3.21}$$

We write that as

$$ {}^t\boldsymbol{F} = {}^t\boldsymbol{R} \tag{3.22}$$

$$ {}^t\boldsymbol{F} = \text{fn}\left({}^tU_1,\ {}^tU_2,\ {}^tU_3 \right) \tag{3.23}$$