2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids	Fall '08
Lecture 4 - Finite element formulation for solids and structures	
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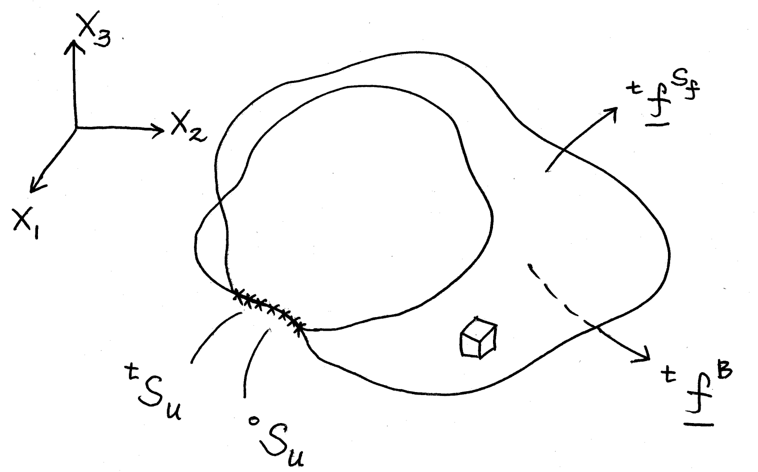
We considered a general 3D body,

Reading: Ch. 4



The exact solution of the mathematical model must satisfy the conditions:

- *Equilibrium* within ${}^tV$ and on ${}^tS_f$,
- *Compatibility*
- *Stress-strain law(s)*

I. Differential formulation

II. Variational formulation (Principle of virtual displacements) (or weak formulation)

We developed the governing F.E. equations for a sheet or bar

We obtained

$$\boxed{{}^t\boldsymbol{F} = {}^t\boldsymbol{R}} \tag{4.1}$$

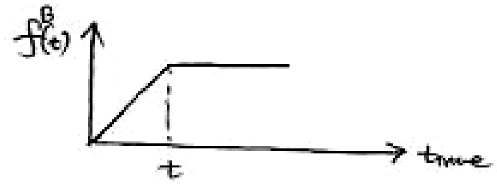
where ${}^t\boldsymbol{F}$ is a function of displacements/stresses/material law; and ${}^t\boldsymbol{R}$ is a function of time.

Assume for now linear analysis: Equilibrium within ${}^0V$ and on ${}^0S_f$, linear stress-strain law and small displacements yields

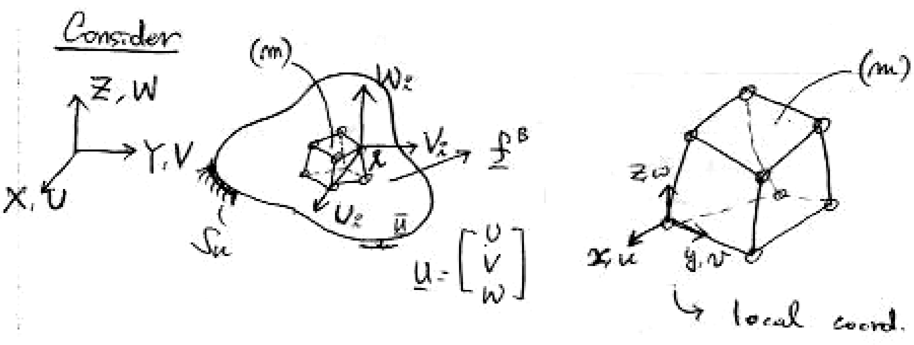
$${}^t\boldsymbol{F} = \boldsymbol{K} \cdot {}^t\boldsymbol{U} \tag{4.2}$$

We want to establish,

$$\boldsymbol{K}\boldsymbol{U}(t) = \boldsymbol{R}(t) \tag{4.3}$$



Consider



$$\hat{\boldsymbol{U}}^T = \left[\begin{array}{cccccc} U_1 & V_1 & W_1 & U_2 & \cdots & W_N \end{array}\right] \quad (N \text{ nodes}) \tag{4.4}$$

where $\hat{\boldsymbol{U}}^T$ is a distinct nodal point displacement vector.

Note: for the moment "remove S_u"

We also say

$$\hat{\boldsymbol{U}}^T = \left[\begin{array}{ccccc} U_1 & U_2 & U_3 & \cdots & U_n \end{array}\right] \quad (n = 3N) \tag{4.5}$$

We now assume

$$\boldsymbol{u}^{(m)} = \boldsymbol{H}^{(m)}\hat{\boldsymbol{U}}, \quad \boldsymbol{u}^{(m)} = \left[\begin{array}{c} u \\ v \\ w \end{array}\right]^{(m)} \tag{4.6a}$$

where $\boldsymbol{H}^{(m)}$ is 3 x n and $\hat{\boldsymbol{U}}$ is n x 1.

$$\boldsymbol{\epsilon}^{(m)} = \boldsymbol{B}^{(m)}\hat{\boldsymbol{U}} \tag{4.6b}$$

where $\boldsymbol{B}^{(m)}$ is 6 x n, and

$$\boldsymbol{\epsilon}^{(m)^T} = \left[\begin{array}{cccccc} \epsilon_{xx} & \epsilon_{yy} & \epsilon_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{array}\right]$$

$$\text{e.g. } \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

We also assume

$$\overline{\boldsymbol{u}}^{(m)} = \boldsymbol{H}^{(m)}\overline{\hat{\boldsymbol{U}}} \tag{4.6c}$$

$$\overline{\boldsymbol{\epsilon}}^{(m)} = \boldsymbol{B}^{(m)}\overline{\hat{\boldsymbol{U}}} \tag{4.6d}$$

Principle of Virtual Work:

$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{\tau} dV = \int_V \overline{\hat{\boldsymbol{U}}}^T \boldsymbol{f}^B dV \tag{4.7}$$

(4.7) can be rewritten as

$$\sum_m \int_{V^{(m)}} \overline{\boldsymbol{\epsilon}}^{(m)^T} \boldsymbol{\tau}^{(m)} dV^{(m)} = \sum_m \int_{V^{(m)}} \overline{\hat{\boldsymbol{U}}}^{(m)^T} \boldsymbol{f}^{B^{(m)}} dV^{(m)} \tag{4.8}$$

Substitute (4.6a) to (4.6d).

$$\begin{aligned} \overline{\hat{\boldsymbol{U}}}^T \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{\tau}^{(m)} \, dV^{(m)} \right\} = \\ \overline{\hat{\boldsymbol{U}}}^T \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B^{(m)}} \, dV^{(m)} \right\} \end{aligned} \tag{4.9}$$

$$\boldsymbol{\tau}^{(m)} = \boldsymbol{C}^{(m)} \boldsymbol{\epsilon}^{(m)} = \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} \hat{\boldsymbol{U}} \tag{4.10}$$

Finally,

$$\begin{aligned} \cancel{\overline{\hat{\boldsymbol{U}}}^T} \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} \, dV^{(m)} \right\} \hat{\boldsymbol{U}} = \\ \cancel{\overline{\hat{\boldsymbol{U}}}^T} \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B^{(m)}} \, dV^{(m)} \right\} \end{aligned} \tag{4.11}$$

with

$$\overline{\boldsymbol{\epsilon}}^{(m)^T} = \overline{\hat{\boldsymbol{U}}}^T \boldsymbol{B}^{(m)^T} \tag{4.12}$$

$$\boldsymbol{K} \hat{\boldsymbol{U}} = \boldsymbol{R}_B \tag{4.13}$$

where $\boldsymbol{K}$ is n x n, and $\boldsymbol{R}_B$ is n x 1.

Direct stiffness method:

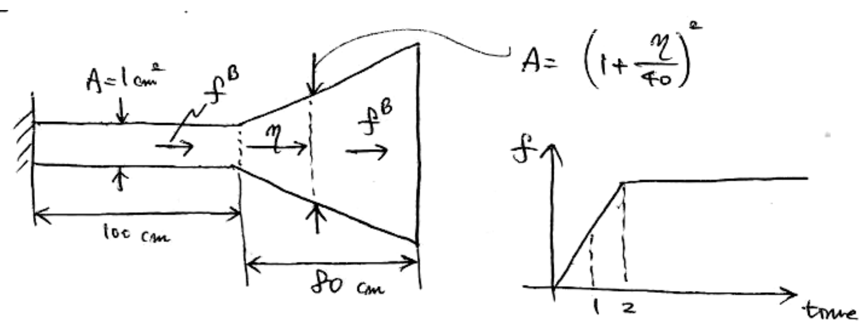
$$\boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)} \tag{4.14}$$

$$\boldsymbol{R}_B = \sum_m \boldsymbol{R}_B^{(m)} \tag{4.15}$$

$$\boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} \, dV^{(m)} \tag{4.16}$$

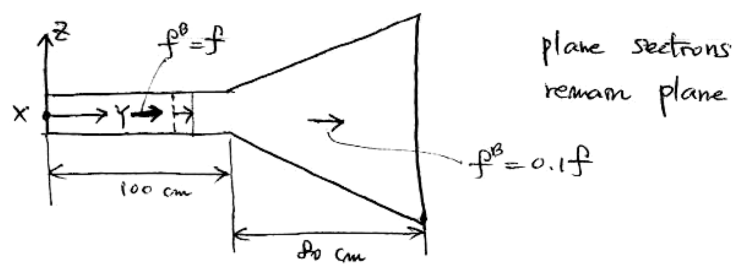
$$\boldsymbol{R}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B^{(m)}} \, dV^{(m)} \tag{4.17}$$

Example 4.5 textbook

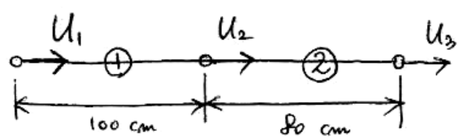


E = Young's Modulus

Mathematical model Plane sections remain plane:

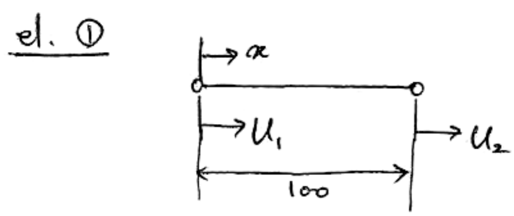


F.E. model

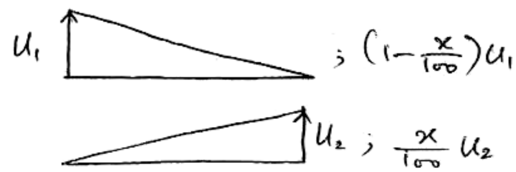


$$\boldsymbol{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{4.18}$$

Element 1



$$u^{(1)}(x) = \underbrace{\begin{bmatrix} 1 - \frac{x}{100} & \frac{x}{100} & 0 \end{bmatrix}}_{\boldsymbol{H}^{(1)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{4.19}$$



$$\epsilon_{xx}^{(1)}(x) = \underbrace{\begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix}}_{\boldsymbol{B}^{(1)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{4.20}$$

Element 2

$$u^{(2)}(x) = \underbrace{\begin{bmatrix} 0 & 1-\frac{x}{80} & \frac{x}{80} \end{bmatrix}}_{\boldsymbol{H}^{(2)}} \boldsymbol{U} \tag{4.21}$$

$$\epsilon_{xx}^{(2)}(x) = \underbrace{\begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix}}_{\boldsymbol{B}^{(2)}} \boldsymbol{U} \tag{4.22}$$

Then,

$$\boldsymbol{K} = \frac{E}{100}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{13E}{240}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{4.23}$$

where,

$$\frac{E(1)}{100} \equiv \left(\frac{AE}{L}\right) \tag{4.24}$$

$$\frac{E\cdot 13}{3\cdot 80} = \underbrace{\left(\frac{13}{3}\right)}_{A^*}\frac{E}{80} \tag{4.25}$$

$$\begin{array}{ccccc} A\Big|_{\eta=0} & < & A^* & < & A\Big|_{\eta=80} \\ 1 & < & 4.333 & < & 9 \end{array} \tag{4.26}$$