2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids	Fall '08

Lecture 5 - F.E. displacement formulation, cont'd

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For the continuum

Reading: Ch. 4

- Differential formulation
- Variational formulation (Principle of Virtual Displacements)

Next, we assumed infinitesimal small displacement, Hooke's Law, linear analysis

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \tag{5.1a}$$

$$\boxed{\boldsymbol{u}^{(m)} = \boldsymbol{H}^{(m)}\boldsymbol{U}} \tag{5.1b}$$

$$\boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)} \tag{5.1c}$$

$$\boldsymbol{R} = \sum_m \boldsymbol{R}_B^{(m)} \tag{5.1d}$$

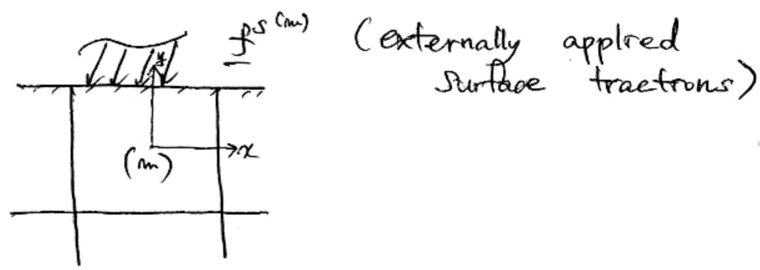
$$\boldsymbol{\epsilon}^{(m)} = \boldsymbol{B}^{(m)}\boldsymbol{U} \tag{5.1e}$$

$$\boldsymbol{U}^T = \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix},\ (n = \text{all d.o.f. of element assemblage}) \tag{5.1f}$$

$$\boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T}\,\boldsymbol{C}^{(m)}\,\boldsymbol{B}^{(m)}\,dV^{(m)} \tag{5.1g}$$

$$\boldsymbol{R}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)^T}\,\boldsymbol{f}^{B^{(m)}}\,dV^{(m)} \tag{5.1h}$$

Surface loads



Recall that in the principle of virtual displacements,

$$\text{“surface” loads} = \int_{S_f} \overline{\boldsymbol{U}}^{S_f\,T}\,\boldsymbol{f}^{S_f}\,dS_f \tag{5.2}$$

$$\boldsymbol{u}^{S^{(m)}} = \boldsymbol{H}^{S^{(m)}}\boldsymbol{U} \tag{5.3}$$

$$\boldsymbol{H}^{S^{(m)}} = \boldsymbol{H}^{(m)}\Big|_{\text{evaluated at the surface}} \tag{5.4}$$

Substitute into (5.2)

$$\overline{\boldsymbol{U}}^T \int_{S^{(m)}} \boldsymbol{H}^{S^{(m)}T} \boldsymbol{f}^{S^{(m)}} \, dS^{(m)} \tag{5.5}$$

for element (m) and one surface of that element.

$$\boldsymbol{R}_s^{(m)} = \int_{S^{(m)}} \boldsymbol{H}^{S^{(m)}T} \boldsymbol{f}^{S^{(m)}} \, dS^{(m)} \tag{5.6}$$

Need to add contributions from all surfaces of all loaded external elements.

$$\boldsymbol{KU} = \boldsymbol{R}_B + \boldsymbol{R}_S + \boldsymbol{R}_c \tag{5.7}$$

where $\boldsymbol{R}_c$ are concentrated nodal loads.

Assume

- (5.7) has been established without any displacement boundary conditions.
- We, however, know nodal displacements $\boldsymbol{U}_b$ (rewriting (5.7)).

$$\boldsymbol{KU} = \boldsymbol{R} \Rightarrow \begin{bmatrix} \boldsymbol{K}_{aa} & \boldsymbol{K}_{ab} \\ \boldsymbol{K}_{ba} & \boldsymbol{K}_{bb} \end{bmatrix} \begin{pmatrix} \boldsymbol{U}_a \\ \boldsymbol{U}_b \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_a \\ \boldsymbol{R}_b \end{pmatrix} \tag{5.8}$$

Solve for $\boldsymbol{U}_a$:

$$\boldsymbol{K}_{aa}\boldsymbol{U}_a = \boldsymbol{R}_a - \boldsymbol{K}_{ab}\boldsymbol{U}_b \tag{5.9}$$

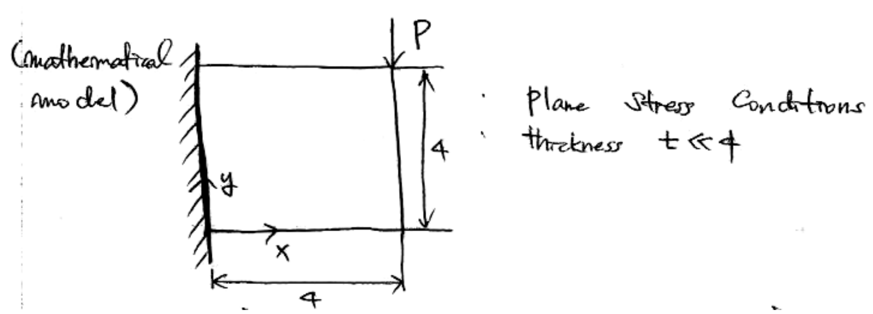
where $\boldsymbol{U}_b$ is known!

Then use

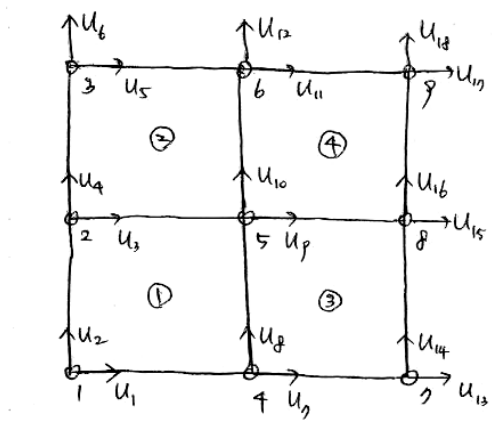
$$\boldsymbol{K}_{ba}\boldsymbol{U}_a + \boldsymbol{K}_{bb}\boldsymbol{U}_b = \boldsymbol{R}_b + \boldsymbol{R}_r \tag{5.10}$$

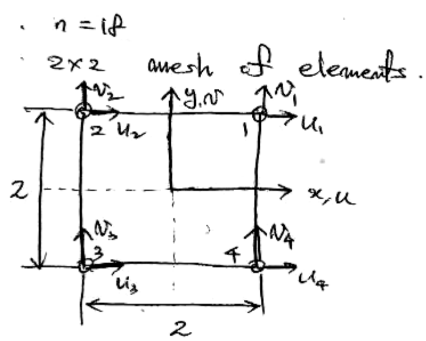
where $\boldsymbol{R}_r$ are unknown reactions.

Example 4.6 textbook



$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} \tag{5.11}$$





$$\left[\begin{array}{c} u(x,y) \\ v(x,y) \end{array}\right] = \boldsymbol{H} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{5.12}$$

If we can set this relation up, then clearly we can get $\boldsymbol{H}^{(1)}$, $\boldsymbol{H}^{(2)}$, $\boldsymbol{H}^{(3)}$, $\boldsymbol{H}^{(4)}$.

$$\boldsymbol{u}^{(m)} = \boldsymbol{H}^{(m)}\boldsymbol{U} \tag{5.13}$$

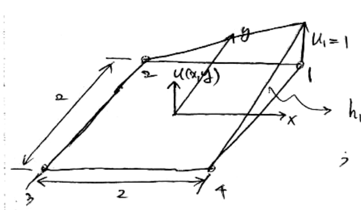
Also want $\boldsymbol{\epsilon}^{(m)} = \boldsymbol{B}^{(m)}\boldsymbol{U}$. *We want* $\boldsymbol{H}$. We could proceed this way

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy \tag{5.14}$$

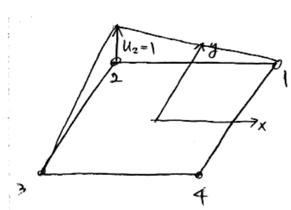
$$v(x,y) = b_1 + b_2 x + b_3 y + b_4 xy \tag{5.15}$$

Express $a_1 \dots a_4$, $b_1 \dots b_4$ in terms of the nodal displacements $u_1 \dots u_4$, $v_1 \dots v_4$.

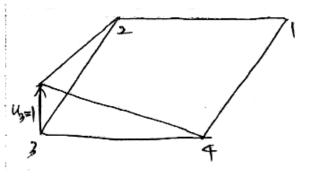
(e.g.) $u(1,1) = a_1 + a_2 + a_3 + a_4 = u_1$.



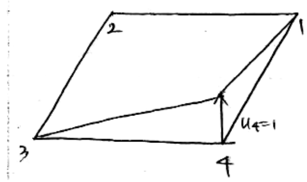
$h_1(x,y) = \frac{1}{4}(1+x)(1+y)$ interpolation function for node 1.



$h_2(x,y) = \frac{1}{4}(1-x)(1+y)$



$$h_3(x,y) = \tfrac{1}{4}(1-x)(1-y)$$



$$h_4(x,y) = \tfrac{1}{4}(1+x)(1-y)$$

$$\boxed{u(x,y) = h_1u_1 + h_2u_2 + h_3u_3 + h_4u_4} \tag{5.16}$$

$$\boxed{v(x,y) = h_1v_1 + h_2v_2 + h_3v_3 + h_4v_4} \tag{5.17}$$

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}}_{\boldsymbol{H}\ (2\text{x}8)} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{5.18}$$

We also want,

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \underbrace{\begin{bmatrix} h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} \\ h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} & h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} \end{bmatrix}}_{\boldsymbol{B}\ (3\text{x}8)} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{5.19}$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \tag{5.20}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \tag{5.21}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5.22}$$