2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 6 - Finite element formulation, example, convergence

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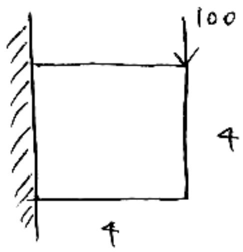
6.1 Example

Reading: Ex. 4.6 in the text



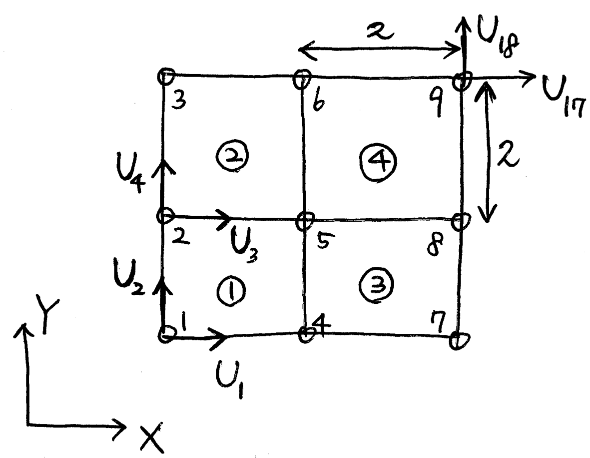
$t = 0.1, \quad E, \nu$ plane stress

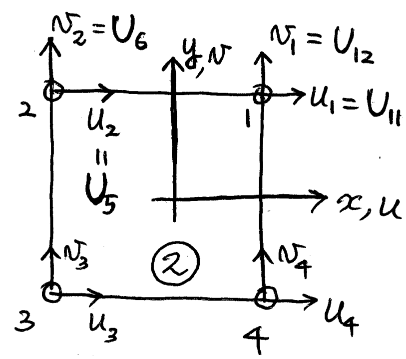
$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}; \quad \boldsymbol{R} = \boldsymbol{R}_B + \boldsymbol{R}_s + \boldsymbol{R}_c + \boldsymbol{R}_r \tag{6.1}$$

$$\boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)}; \quad \boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{C}^{(m)} \, \boldsymbol{B}^{(m)} \, d\, V^{(m)} \tag{6.2}$$

$$\boldsymbol{R}_B = \sum_m \boldsymbol{R}_B^{(m)}; \quad \boldsymbol{R}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B(m)} \, d\, V^{(m)} \tag{6.3}$$

6.1.1 F.E. model





$$\boldsymbol{K}\Big|_{\text{el. (2)}} = \begin{array}{c} \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & v_1 & v_2 & v_3 & v_4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} \bigcirc & \square & \triangle & \times & \times & \times & \times & \times \\ \vdots & & & & & & & \\ & & & & & & & \end{bmatrix} \end{array} \begin{array}{l} \leftarrow u_1 \\ \vdots \\ \\ \end{array} \tag{6.4}$$

In practice,

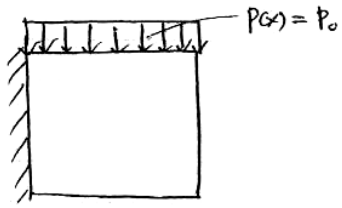
$$\boldsymbol{K}\Big|_{\text{el}} = \int_V \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} \, dV; \quad \boldsymbol{\epsilon} = \boldsymbol{B} \begin{pmatrix} u_1 \\ \vdots \\ u_4 \\ v_1 \\ \vdots \\ v_4 \end{pmatrix} \tag{6.5}$$

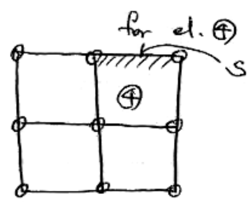
where $\boldsymbol{K}$ is 8x8 and $\boldsymbol{B}$ is 3x8.

Assume we have $\boldsymbol{K}$ (8x8) for el. (2)

$$\underbrace{\boldsymbol{K}}_{\text{assemblage}} = \begin{array}{c} \begin{array}{ccccccccc} U_1 & U_2 & U_3 & U_4 & U_5 & \cdots & U_{11} & \cdots & U_{18} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{ccccccccc} \times & \times & \times & \times & \times & \times & \times & \times & \times \\ & & \vdots & & \vdots & & \vdots & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \triangle & & \square & & \bigcirc & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \vdots & & \vdots & & \vdots & & \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{array} \right] \end{array} \begin{array}{l} \leftarrow U_1 \\ \vdots \\ \vdots \\ \leftarrow U_{11} \\ \vdots \\ \vdots \\ \leftarrow U_{18} \end{array} \tag{6.6}$$

Consider,





$$\boldsymbol{R}_S = \int_S \boldsymbol{H}^{S^T} \boldsymbol{f}^S \, d\, S; \quad \boldsymbol{H}^S = \boldsymbol{H}\Big|_{\text{on surface}} \tag{6.7}$$

$$\boldsymbol{H} = \left[\begin{array}{cccccccc} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{array} \right] \begin{array}{l} \leftarrow u(x,y) \\ \leftarrow v(x,y) \end{array} \tag{6.8}$$

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_4 \\ v_1 \\ \vdots \\ v_4 \end{pmatrix} \tag{6.9}$$

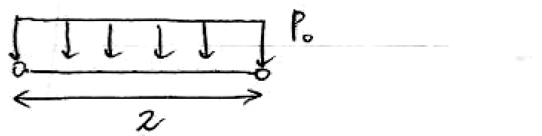
$$\boldsymbol{H}^S = \boldsymbol{H}\Big|_{y=+1} \tag{6.10}$$

$$= \begin{bmatrix} \frac{1}{2}(1+x) & \frac{1}{2}(1-x) & 0 & 0 & 0 & 0 & 0 & 0 \\ \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1+x) & \frac{1}{2}(1-x) & 0 & 0 \end{bmatrix} \tag{6.11}$$

From (6.7);

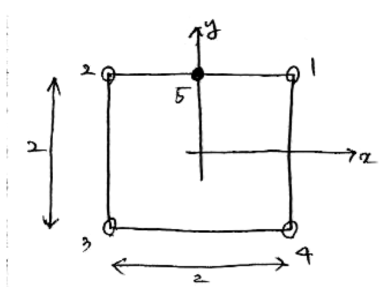
$$\boldsymbol{R}_S = \int_{-1}^{+1} \left[\begin{array}{c|c} \frac{1}{2}(1+x) & 0 \\ \frac{1}{2}(1-x) & 0 \\ 0 & \frac{1}{2}(1+x) \\ 0 & \frac{1}{2}(1-x) \end{array}\right] \begin{bmatrix} 0 \\ -p(x) \end{bmatrix} \underbrace{(0.1)}_{\text{thickness}} \; dx \tag{6.12}$$

$$\boldsymbol{R}_S = \begin{bmatrix} 0 \\ 0 \\ -p_0(0.1) \\ -p_0(0.1) \end{bmatrix} \tag{6.13}$$



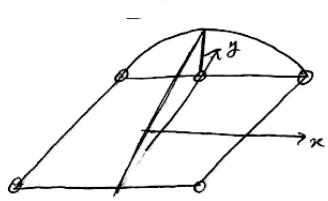
total load = P₀ × 0.1 × 2

6.1.2 Higher-order elements



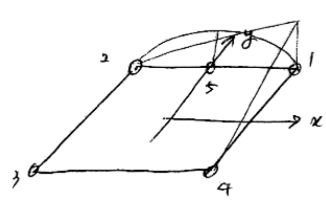
Want h_1, h_2, h_3, h_4, h_5

$u(x,y) = \sum_{i=1}^{5} h_i u_i$.



$h_i = 1$ at node i and 0 at all other nodes.

$h_5 = \frac{1}{2}(1-x^2)(1+y)$



$$h_1 = \frac{1}{4}(1+x)(1+y) - \frac{1}{2}h_5 \tag{6.14}$$

$$h_2 = \frac{1}{4}(1-x)(1+y) - \frac{1}{2}h_5 \tag{6.15}$$

$$h_3 = \frac{1}{4}(1-x)(1-y) \tag{6.16}$$

$$h_1 = \frac{1}{4}(1+x)(1-y) \tag{6.17}$$

Note:

$$\boxed{\sum h_i = 1}$$

We must have $\sum_i h_i = 1$ to satisfy the rigid body mode condition.

$$u(x,y) = \sum_i h_i u_i \tag{6.18}$$

Assume all nodal point displacements $= u^*$. Then,

$$u(x,y) = \sum_i h_i u^* = u^* \sum_i h_i = u^* \tag{6.19}$$

From (6.1),

$$\left(\sum_m \boldsymbol{K}^{(m)}\right)\boldsymbol{U} = \boldsymbol{R} \tag{6.20}$$

$$\sum_m \left[\int_{V^{(m)}} \boldsymbol{B}^{(m)^T}\boldsymbol{C}^{(m)}\boldsymbol{B}^{(m)} dV^{(m)}\right]\boldsymbol{U} = \boldsymbol{R} \tag{6.21}$$

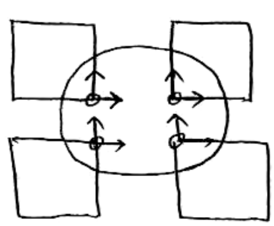
where $\boldsymbol{C}^{(m)}\boldsymbol{B}^{(m)}\boldsymbol{U} = \boldsymbol{\tau}^{(m)}$. (Assume we calculated $\boldsymbol{U}$.)

$$\sum_m \int_{V^{(m)}} \boldsymbol{B}^{(m)^T}\boldsymbol{\tau}^{(m)}\, d\, V^{(m)} = \boldsymbol{R} \tag{6.22}$$

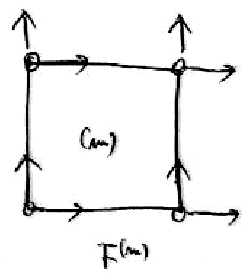
$$\sum_m \boldsymbol{F}^{(m)} = \boldsymbol{R}; \quad \boldsymbol{F}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T}\boldsymbol{\tau}^{(m)}\, d\, V^{(m)} \tag{6.23}$$

Two properties

I. The sum of the $\boldsymbol{F}^{(m)}$'s at any node is equal to the applied external forces.



II. Every element is in equilibrium under its $\boldsymbol{F}^{(m)}$



$$\bar{\hat{\boldsymbol{U}}}^T \boldsymbol{F}^{(m)} = \underbrace{\bar{\hat{\boldsymbol{U}}}^T \int_{V^{(m)}} \boldsymbol{B}^{(m)^T}}_{=\overline{\boldsymbol{\epsilon}}^{(m)^T}} \boldsymbol{\tau}^{(m)} \, d\, V^{(m)} \tag{6.24}$$

$$= \int_{V^{(m)}} \overline{\boldsymbol{\epsilon}}^{(m)^T} \boldsymbol{\tau}^{(m)} \, d\, V^{(m)} \tag{6.25}$$

$$= 0 \tag{6.26}$$

where $\bar{\hat{\boldsymbol{U}}}^T$ = virtual nodal point displacement.

Apply rigid body displacement.

If we move the element virtually in the rigid body modes, $\overline{\boldsymbol{\epsilon}}^{(m)}$ is zero. Therefore the virtual work obtained due to virtual motion of the element is zero. Then the element is in equilibrium under its $\boldsymbol{F}^{(m)}$.