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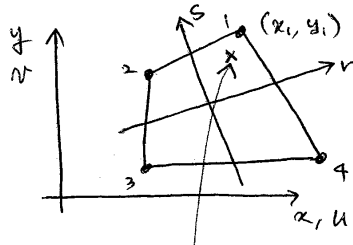
2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

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Lecture 7 - Isoparametric elements

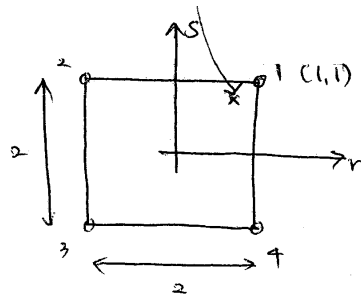
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We want $K = \int_V B^T C B dV$, $R_B = \int_V H^T f^B dV$. Unique correspondence $(x, y) \Leftrightarrow (r, s)$

Reading:
Sec. 5.1-5.3



(r, s) are natural coordinate system or isoparametric coordinate system.

$$x = \sum_{i=1}^4 h_i x_i \quad (7.1)$$

$$y = \sum_{i=1}^4 h_i y_i \quad (7.2)$$

where

$$h_1 = \frac{1}{4}(1+r)(1+s) \quad (7.3)$$

$$h_2 = \frac{1}{4}(1-r)(1+s) \quad (7.4)$$

...

$$u(r, s) = \sum_{i=1}^4 h_i u_i \quad (7.5)$$

$$v(r, s) = \sum_{i=1}^4 h_i v_i \quad (7.6)$$

$$\boldsymbol{\epsilon} = \mathbf{B}\hat{\mathbf{u}} \quad \hat{\mathbf{u}}^T = [u_1 \quad u_2 \quad \cdots \quad v_4] \quad (7.7)$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \mathbf{B}\hat{\mathbf{u}} \quad (7.8)$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}}_{\mathbf{J}} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \quad (7.9)$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} \quad (7.10)$$

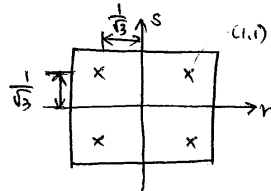
\mathbf{J} must be non-singular which ensures that there is unique correspondence between (x, y) and (r, s) . Hence,

$$\mathbf{K} = \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{C} \mathbf{B} \underbrace{t \det(\mathbf{J})}_{dV} dr ds \quad (7.11)$$

$$\text{Also, } \mathbf{R}_B = \int_{-1}^1 \int_{-1}^1 \mathbf{H}^T \mathbf{f}^B t \det(\mathbf{J}) dr ds \quad (7.12)$$

Numerical integration (Gauss formulae) (Ch. 5.5)

$$\mathbf{K} \cong t \sum_i \sum_j \mathbf{B}_{ij}^T \mathbf{C} \mathbf{B}_{ij} \det(\mathbf{J}_{ij}) \times (\text{weight } i, j) \quad (7.13)$$

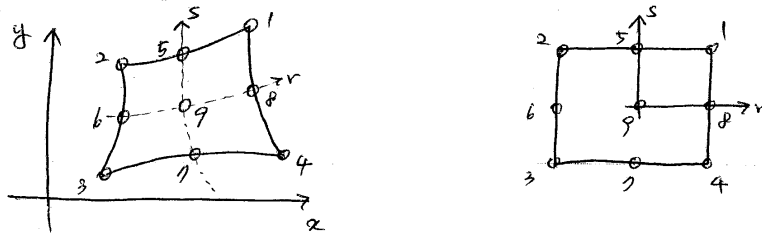


2x2 Gauss integration,

$$(i = 1, 2) \quad (7.14)$$

$$(j = 1, 2) \quad (\text{weight } i, j = 1 \text{ in this case}) \quad (7.15)$$

9-node element



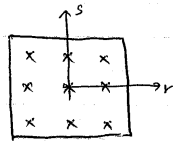
$$x = \sum_{i=1}^9 h_i x_i \tag{7.16}$$

$$y = \sum_{i=1}^9 h_i y_i \tag{7.17}$$

$$u = \sum_{i=1}^9 h_i u_i \tag{7.18}$$

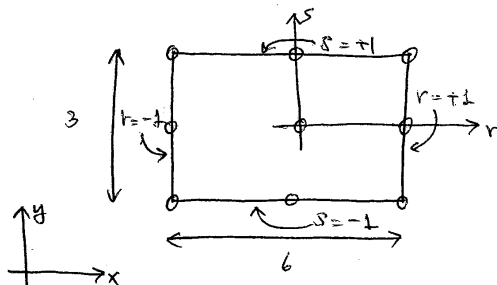
$$v = \sum_{i=1}^9 h_i v_i \tag{7.19}$$

Use 3x3 Gauss integration



For rectangular elements, $\mathbf{J} = \text{const}$

Consider the following element,



Note, here we could use $h_i(x, y)$ directly.

$$\mathbf{J} = \begin{bmatrix} 3 \left(= \frac{6}{2} \right) & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \tag{7.20}$$

Then, we can determine the number of appropriate integration points by investigating the maximum order of $\mathbf{B}^T \mathbf{C} \mathbf{B}$.

For a rectangular element, 3x3 Gauss integration gives exact \mathbf{K} matrix. If the element is distorted, a \mathbf{K} matrix which is still accurate enough will be obtained, (if high enough integration is used).

Convergence *Principle of virtual work:*

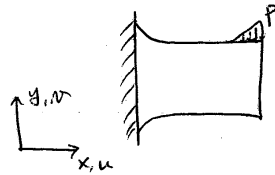
$$\int_V \bar{\epsilon}^T C \epsilon dV = \mathcal{R}(\bar{u}) \quad (7.21)$$

Reading:
Sec. 5.5.5,
4.3

Find \mathbf{u} , solution, in V , vector space (any continuous function that satisfies boundary conditions), satisfying

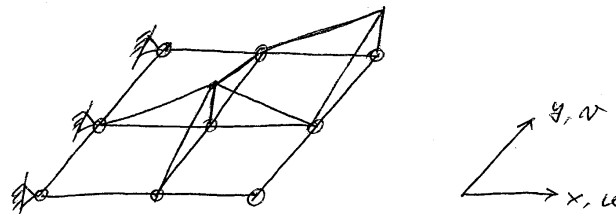
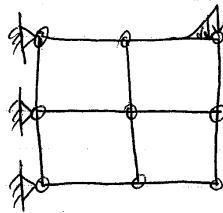
$$\int_V \bar{\epsilon}^T C \epsilon dV = \underbrace{a(\mathbf{u}, \mathbf{v})}_{\text{bilinear form}} = \underbrace{(\mathbf{f}, \mathbf{v})}_{\mathcal{R}(\mathbf{v})} \quad \text{for all } \mathbf{v}, \text{ an element of } V. \quad (7.22)$$

Example:



Finite Element problem Find $\mathbf{u}_h \in V_h$, where V_h is F.E. vector space such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h \quad (7.23)$$



Size of $V_h \Rightarrow \#$ of independent DOFs (here it's 12).

Note:

$$\underbrace{a(\mathbf{w}, \mathbf{w})}_{2x \text{ (strain energy when imposing } \mathbf{w})} > 0 \text{ for } \mathbf{w} \in V \quad (\mathbf{w} \neq \mathbf{0})$$

Also,

$$a(\mathbf{w}_h, \mathbf{w}_h) > 0 \text{ for } \mathbf{w}_h \in V_h \quad (V_h \subset V, \mathbf{w}_h \neq \mathbf{0})$$

Property I Define: $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$.

$$\text{From (7.22), } a(\mathbf{u}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad (7.24)$$

$$\text{From (7.23), } a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad (7.25)$$

Hence,

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0 \quad (7.26)$$

$$a(\mathbf{e}_h, \mathbf{v}_h) = 0 \quad (7.27)$$

(error is orthogonal in that sense to all \mathbf{v}_h in F.E. space).

Property II

$$\boxed{a(\mathbf{u}_h, \mathbf{u}_h) \leq a(\mathbf{u}, \mathbf{u})} \quad (7.28)$$

Proof:

$$a(\mathbf{u}, \mathbf{u}) = a(\mathbf{u}_h + \mathbf{e}_h, \mathbf{u}_h + \mathbf{e}_h) \quad (7.29)$$

$$= a(\mathbf{u}_h, \mathbf{u}_h) + \underbrace{2a(\mathbf{u}_h, \mathbf{e}_h)}_{\substack{\rightarrow 0 \text{ by Prop. I} \\ \geq 0}} + \underbrace{a(\mathbf{e}_h, \mathbf{e}_h)}_{\geq 0} \quad (7.30)$$

$$\therefore a(\mathbf{u}, \mathbf{u}) \geq a(\mathbf{u}_h, \mathbf{u}_h) \quad (7.31)$$

