2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

Lecture 8 - Convergence of displacement-based FEM

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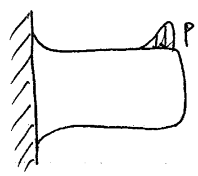
(A) Find

$$\boldsymbol{u} \in \mathrm{V} \text{ such that } a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathrm{V} \text{ (Mathematical model)} \tag{8.1}$$

$$a(\boldsymbol{v}, \boldsymbol{v}) > 0 \quad \forall \boldsymbol{v} \in \mathrm{V}, \quad \boldsymbol{v} \neq \boldsymbol{0}. \tag{8.2}$$

where (8.2) implies that structures are supported properly. E.g.



(B) F.E. Problem Find

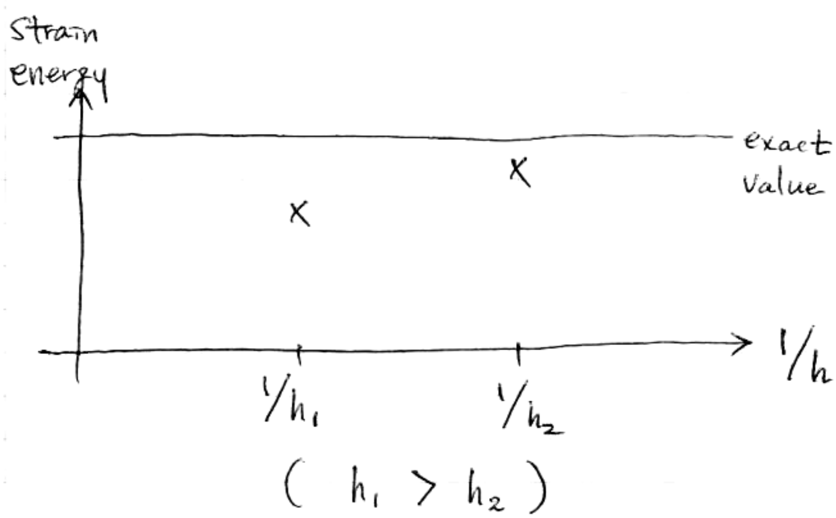
$$\boldsymbol{u}_h \in \mathrm{V}_h \text{ such that } a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h \tag{8.3}$$

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h) > 0 \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h, \quad \boldsymbol{v}_h \neq 0 \tag{8.4}$$

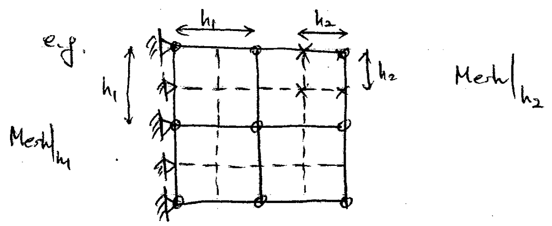
Properties $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$

(I) $a(\boldsymbol{e}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h$ (8.5)

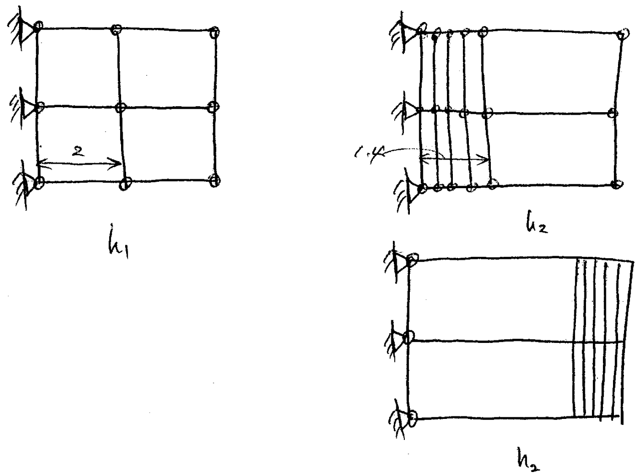
(II) $a(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq a(\boldsymbol{u}, \boldsymbol{u})$ (8.6)



(C) Assume $\text{Mesh}\Big|_{h_1}$ "is contained in" $\text{Mesh}\Big|_{h_2}$



e.g. $\text{Mesh}\Big|_{h_1}$ not contained in $\text{Mesh}\Big|_{h_2}$



We assume (C), but need another property (independent of (C))

$$\textbf{(III)}\ a(\boldsymbol{e}_h, \boldsymbol{e}_h) \leq a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{u} - \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h \tag{8.7}$$

$\boldsymbol{u}_h$ minimizes! (Recall $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$)

Proof: Pick $\boldsymbol{w}_h \in \mathrm{V}_h$.

$$a(\boldsymbol{e}_h + \boldsymbol{w}_h, \boldsymbol{e}_h + \boldsymbol{w}_h) = a(\boldsymbol{e}_h, \boldsymbol{e}_h) + \cancelto{0}{2a(\boldsymbol{e}_h, \boldsymbol{w}_h)} + \underbrace{a(\boldsymbol{w}_h, \boldsymbol{w}_h)}_{\geq 0} \tag{8.8}$$

Equality holds for ($\boldsymbol{w}_h = \boldsymbol{0}$)

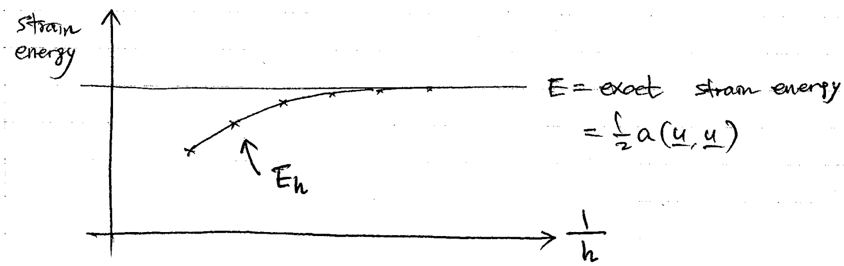
$$a(\boldsymbol{e}_h, \boldsymbol{e}_h) \leq a(\boldsymbol{e}_h + \boldsymbol{w}_h, \boldsymbol{e}_h + \boldsymbol{w}_h) \tag{8.9}$$

$$= a(\boldsymbol{u} - \boldsymbol{u}_h + \boldsymbol{w}_h, \boldsymbol{u} - \boldsymbol{u}_h + \boldsymbol{w}_h) \tag{8.10}$$

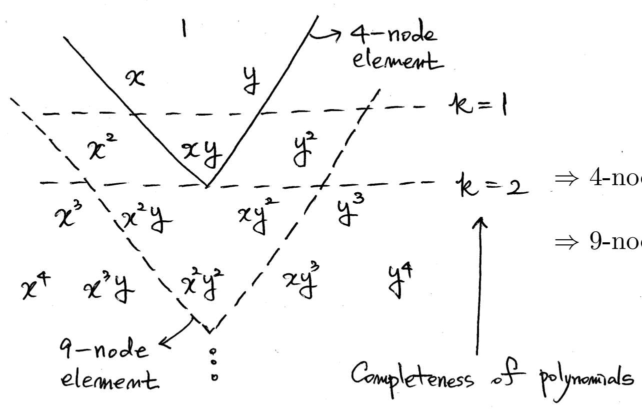
Take $\boldsymbol{w}_h = \boldsymbol{u}_h - \boldsymbol{v}_h$.

$$a(\boldsymbol{e}_h, \boldsymbol{e}_h) \leq a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{u} - \boldsymbol{v}_h) \tag{8.11}$$

Using property (III) and (C), we can say that we will converge monotonically, from below, to $a(\boldsymbol{u}, \boldsymbol{u})$:



Pascal triangle (2D)



⇒ 4-node element is complete to $k = 1$.

⇒ 9-node element is complete to $k = 2$.

(Ch. 4.3)

$$\text{error in displacement} \sim C \cdot h^{k+1} \tag{8.12}$$

(C is a constant determined by the exact solution, material property...)

$$\text{error in stresses} \sim C \cdot h^{k} \tag{8.13}$$

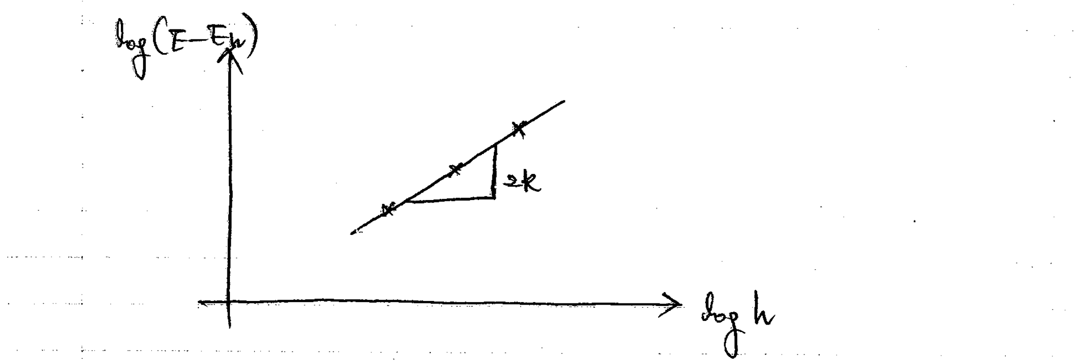
$$\boxed{\text{error in strain energy} \sim C \cdot h^{2k}} \quad (\leftarrow \text{these } C \text{ are different}) \tag{8.14}$$

Hence,

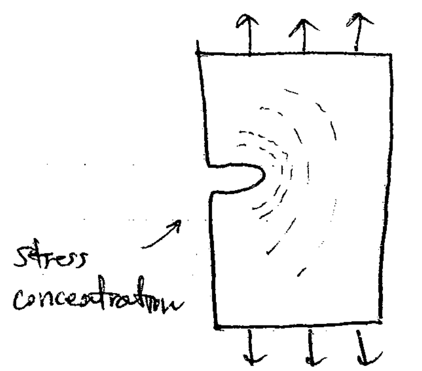
$$E - E_h = C \cdot h^{2k} \quad \text{(roughly equal to)} \tag{8.15}$$

By theory,

$$\log(E - E_h) = \log C + 2k \log h \tag{8.16}$$



By experiment, we can evaluate $\log(E - E_h)$ for different meshes and plot $\log(E - E_h)$ vs. $\log h$



We need to use graded meshes if we have high stress gradients.

Example Consider an almost incompressible material:

$$\epsilon_V = \text{ vol. strain} \tag{8.17}$$

or

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} \rightarrow \text{ very small or zero} \tag{8.18}$$

We can "see" difficulties:

$$p = -\kappa \epsilon_V \quad \kappa = \text{ bulk modulus} \tag{8.19}$$

As the material becomes incompressible ($\nu = 0.3 \rightarrow 0.4999$)

$$\left.\begin{aligned} \kappa &\rightarrow \infty \\ \epsilon_V &\rightarrow 0 \end{aligned}\right\} \quad p \rightarrow \text{ finite number} \tag{8.20}$$

(Small error in ϵ_V results in huge error on pressure as $\kappa \rightarrow \infty$, the constant C in (8.15) can be very large $\Rightarrow$ locking)