2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

Lecture 17 - Incompressible fluid flow and heat transfer, cont'd

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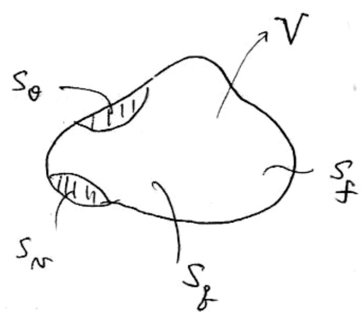
17.1 Abstract body

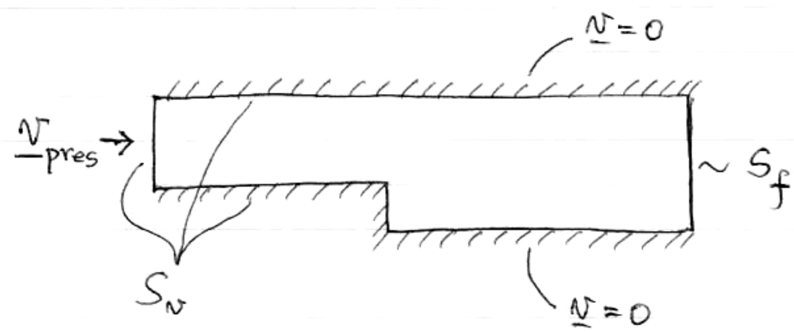
Reading:
Sec. 7.4

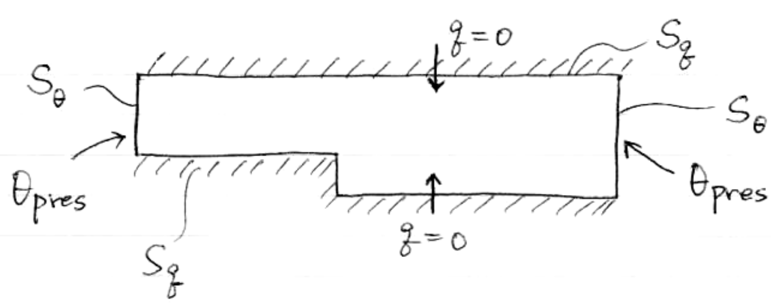


Fluid Flow
S_v, S_f
$S_v \cup S_f = S$
$S_v \cap S_f = 0$

Heat transfer
S_θ, S_q
$S_\theta \cup S_q = S$
$S_\theta \cap S_q = 0$

17.2 Actual 2D problem (channel flow)





17.3 Basic equations

P.V. velocities

$$\int_V \overline{v}_i \rho v_{i,j} v_j dV + \int_V \tau_{ij} \overline{e}_{ij} dV = \int_V \overline{v}_i f_i^B dV + \int_{S_f} \overline{v}_i^{S_f} f_i^{S_f} dS_f \tag{17.1}$$

Continuity

$$\int_V \overline{p} v_{i,i} dV = 0 \tag{17.2}$$

P.V. temperature

$$\int_V \overline{\theta} \rho c_p \theta_{,i} v_i dV + \int_V \overline{\theta}_{,i} k \theta_{,i} dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^S q^S dS \tag{17.3}$$

F.E. solution

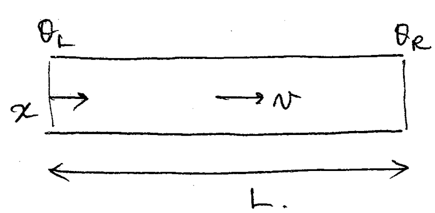
$$x_i = \sum h_k x_i^k \tag{17.4}$$
$$v_i = \sum h_k v_i^k \tag{17.5}$$
$$\theta = \sum h_k \theta_k \tag{17.6}$$
$$p = \sum \tilde{h}_k p_k \tag{17.7}$$

$$\Rightarrow \boxed{\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{R}} \qquad \boldsymbol{u} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{p} \\ \boldsymbol{\theta} \end{pmatrix} \text{ nodal variables} \tag{17.8}$$

17.4 Model problem

1D equation,

$$\rho c_p v \frac{d\theta}{dx} = k \frac{d^2\theta}{dx^2} \tag{17.9}$$

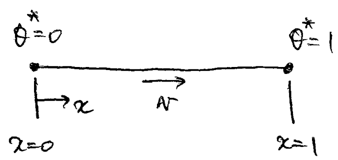


(v is given, unit cross section)

Non-dimensional form (Section 7.4)

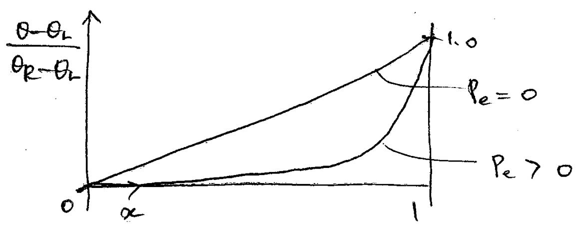
$$\boxed{\text{Pe} \frac{d\theta}{dx} = \frac{d^2\theta}{dx^2}} \tag{17.10}$$

$$\mathrm{Pe} = \frac{vL}{\alpha}, \qquad \alpha = \frac{k}{\rho c_p} \tag{17.11}$$



θ^* is non-dimensional

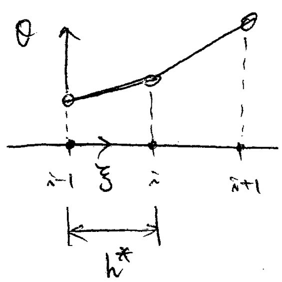
$$\frac{\theta - \theta_L}{\theta_R - \theta_L} = \frac{\exp\left(\frac{\mathrm{Pe}\, x}{L}\right) - 1}{\exp\left(\mathrm{Pe}\right) - 1} \tag{17.12}$$



(17.10) in F.E. analysis becomes

$$\int_V \overline{\theta} \mathrm{Pe} \frac{d\theta}{dx} dV + \int_V \frac{d\overline{\theta}}{dx}\frac{d\theta}{dx} dV = 0 \tag{17.13}$$

Discretized by linear elements:



$$h^* = \frac{h}{L}$$

$$\theta(\xi) = \left(1 - \frac{\xi}{h^*}\right)\theta_{i-1} + \frac{\xi}{h^*}\theta_i \tag{17.14}$$

For node i:

$$-\theta_{i-1} - \frac{\mathrm{Pe}^e}{2}\theta_{i-1} + 2\theta_i - \theta_{i+1} + \frac{\mathrm{Pe}^e}{2}\theta_{i+1} = 0 \tag{17.15}$$

where

$$\mathrm{Pe}^e = \frac{vh}{\alpha} \qquad \left(= \mathrm{Pe}\frac{h}{L}\right) \tag{17.16}$$

This result is the same as obtained by finite differences

$$\theta''\Big|_i = \frac{1}{(h^*)^2}\left(\theta_{i+1} - 2\theta_i + \theta_{i-1}\right) \tag{17.17}$$

$$\theta'\Big|_i = \frac{\theta_{i+1} - \theta_{i-1}}{2h^*} \tag{17.18}$$

Considered $\theta_{i+1} = 1$, $\theta_{i-1} = 0$. Then

$$\theta_i = \frac{1 - (\mathrm{Pe}^e/2)}{2} \tag{17.19}$$

Physically unrealistic solution when $\mathrm{Pe}^e > 2$. For this not to happen, we should refine the mesh—a very fine mesh would be required. We use "upwinding"

$$\left.\frac{d\theta}{dx}\right|_i = \frac{\theta_i - \theta_{i-1}}{h^*} \tag{17.20}$$

The result is

$$\left(-1 - \mathrm{Pe}^e\right)\theta_{i-1} + \left(2 + \mathrm{Pe}^e\right)\theta_i - \theta_{i+1} = 0 \tag{17.21}$$

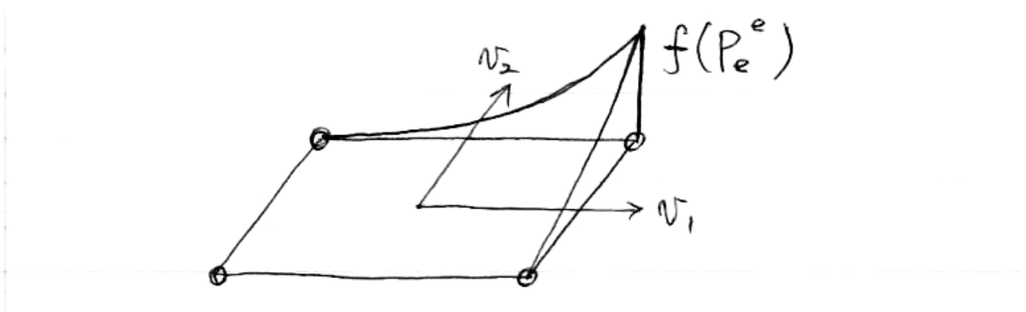
Very stable, e.g.

$$\left.\begin{aligned}\theta_{i-1} &= 0\\ \theta_{i+1} &= 1\end{aligned}\right\} \Rightarrow \theta_i = \frac{1}{2 + \mathrm{Pe}^e} \tag{17.22}$$

Unfortunately it is not that accurate. To obtain better accuracy in the interpolation for θ, use the function

$$\frac{\exp\left(\mathrm{Pe}\frac{x}{L}\right) - 1}{\exp\left(\mathrm{Pe}\right) - 1} \tag{17.23}$$

The result is Pe^e dependent:



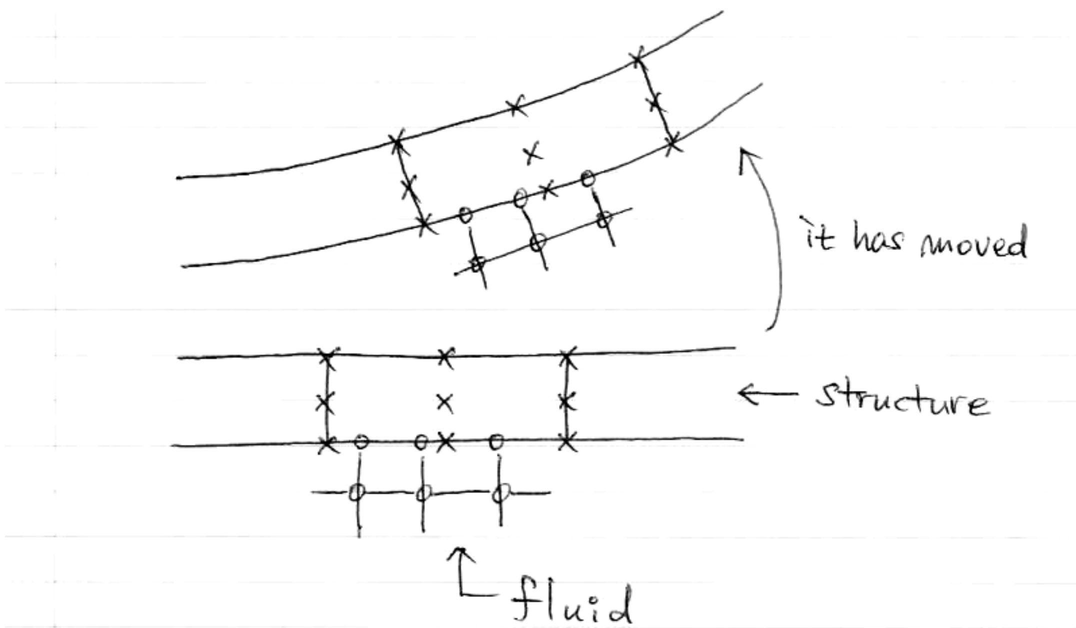
This implies flow-condition based interpolation. We use such interpolation functions—see references.

References

[1] K.J. Bathe and H. Zhang. "A Flow-Condition-Based Interpolation Finite Element Procedure for Incompressible Fluid Flows." *Computers & Structures*, 80:1267–1277, 2002.

[2] H. Kohno and K.J. Bathe. "A Flow-Condition-Based Interpolation Finite Element Procedure for Triangular Grids." *International Journal for Numerical Methods in Fluids*, 51:673–699, 2006.

17.5 FSI briefly



Lagrangian formulation for the structure/solid

Arbitrary Lagrangian-Eulerian (ALE) formulation Let f be a variable of a particle (e.g. $f = \theta$). Consider 1D

$$\dot{f}\Big|_{\text{particle}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v \tag{17.24}$$

where v is the particle velocity. For a mesh point,

$$f^*\Big|_{\text{mesh point}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v_m \tag{17.25}$$

where v_m is the mesh point velocity. Hence,

$$\dot{f}\Big|_{\text{particle}} = f^*\Big|_{\text{mesh point}} + \frac{\partial f}{\partial x}(v - v_m) \tag{17.26}$$

Use (17.26) in the momentum and energy equations and use force equilibrium and compatibility at the FSI boundary to set up the governing F.E. equations.

References

[1] K.J. Bathe, H. Zhang and M.H. Wang. "Finite Element Analysis of Incompressible and Compressible Fluid Flows with Free Surfaces and Structural Interactions." *Computers & Structures*, 56:193–213, 1995.

[2] K.J. Bathe, H. Zhang and S. Ji. "Finite Element Analysis of Fluid Flows Fully Coupled with Structural Interactions." *Computers & Structures*, 72:1–16, 1999.

[3] K.J. Bathe and H. Zhang. "Finite Element Developments for General Fluid Flows with Structural Interactions." *International Journal for Numerical Methods in Engineering*, 60:213–232, 2004.