2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 2 - Finite element formulation of solids and structures

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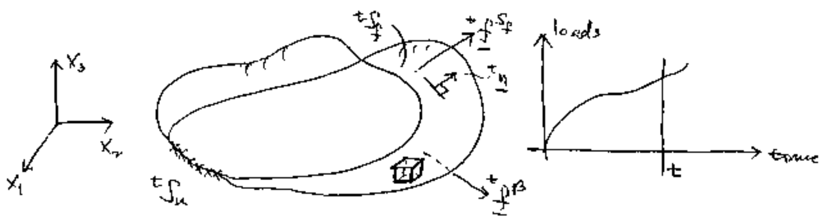


Reading: Ch. 1, Sec. 6.1-6.2

Assume that on ${}^tS_u$ the displacements are zero (and ${}^tS_u$ is constant). Need to satisfy at time t:

- *Equilibrium* of Cauchy stresses ${}^t\tau_{ij}$ with applied loads

$$ {}^t\tau^T = \left[\ {}^t\tau_{11} \quad {}^t\tau_{22} \quad {}^t\tau_{33} \quad {}^t\tau_{12} \quad {}^t\tau_{23} \quad {}^t\tau_{31} \ \right] \tag{2.1} $$

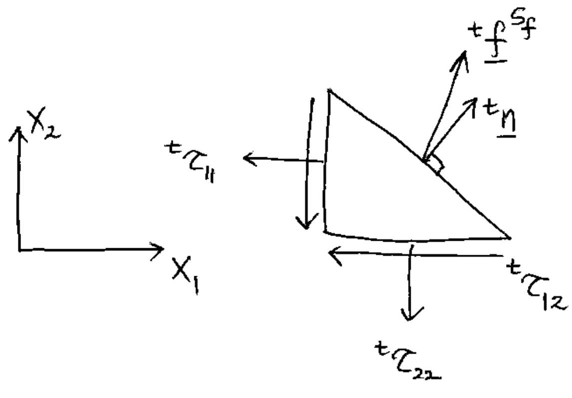
(For $i = 1, 2, 3$)

$$ {}^t\tau_{ij,j} + {}^tf_i^B = 0 \text{ in } {}^tV \text{ (sum over } j) \tag{2.2} $$

$$ {}^t\tau_{ij}\,{}^tn_j = {}^tf_i^{S_f} \text{ on } {}^tS_f \text{ (sum over } j) \tag{2.3} $$

$$ (\text{e.g. } {}^tf_i^{S_f} = {}^t\tau_{i1}\,{}^tn_1 + {}^t\tau_{i2}\,{}^tn_2 + {}^t\tau_{i3}\,{}^tn_3\ ) \tag{2.4} $$

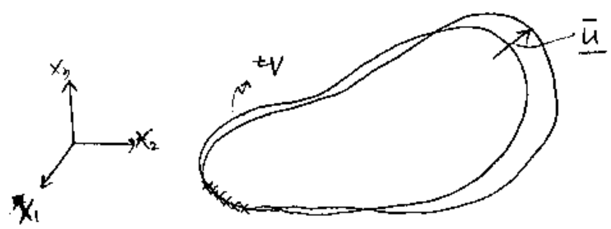
And: ${}^t\tau_{11}\,{}^tn_1 + {}^t\tau_{12}\,{}^tn_2 = {}^tf_1^{S_f}$



- *Compatibility* The displacements ${}^tu_i$ need to be continuous and zero on ${}^tS_u$.
- *Stress-Strain law*

$$ {}^t\tau_{ij} = \text{ function}\left({}^tu_j\right) \tag{2.5} $$

2.1 Principle of Virtual Work*



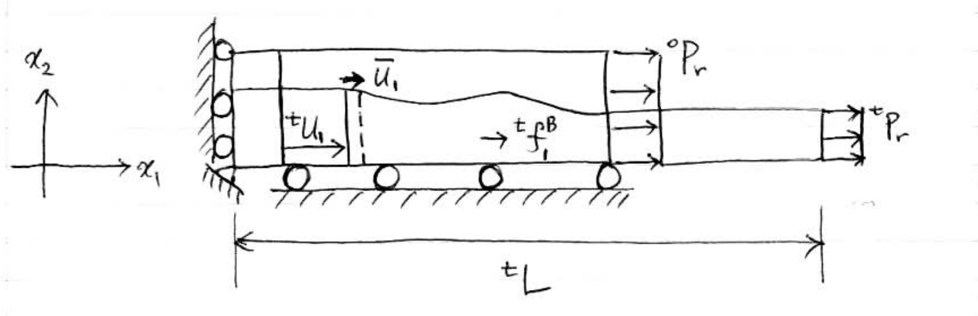
$$\int_{{}^tV} {}^t\tau_{ij}\, {}_t\overline{e}_{ij}\, d\,{}^tV = \int_{{}^tV} {}^tf_i^B\, \overline{u}_i\, d\,{}^tV + \int_{{}^tS_f} {}^tf_i^{S_f}\, \overline{u}_i^{S_f}\, d\,{}^tS_f \tag{2.6}$$

where

$$ {}_t\overline{e}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial\, {}^tx_j} + \frac{\partial \overline{u}_j}{\partial\, {}^tx_i}\right) \tag{2.7}$$

$$\text{with } \overline{u}_i\Big|_{{}^tS_u} = 0 \tag{2.8}$$

2.2 Example



Assume "plane sections remain plane"

Principle of Virtual Work

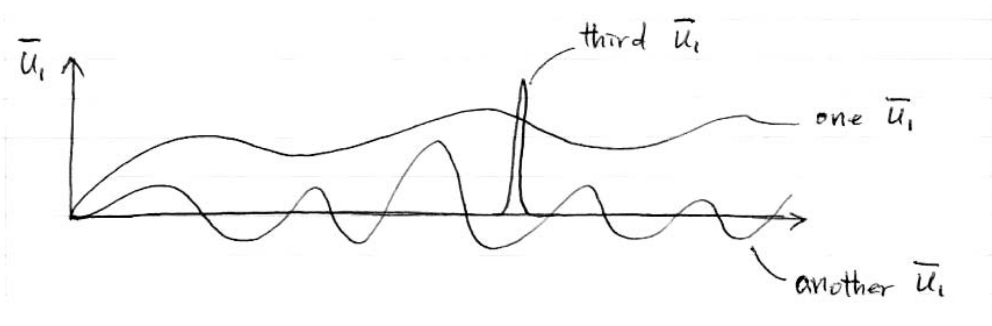
$$\int_{{}^tV} {}^t\tau_{11}\, {}_t\overline{e}_{11}\, d\,{}^tV = \int_{{}^tV} {}^tf_1^B\, \overline{u}_1\, d\,{}^tV + \int_{{}^tS_f} {}^tP_r\, \overline{u}_1^{S_f}\, d\,{}^tS_f \tag{2.9}$$

Derivation of (2.9)

$$ {}^t\tau_{11,1} + {}^tf_1^B = 0 \quad \text{by (2.2)} \tag{2.10}$$

$$\left({}^t\tau_{11,1} + {}^tf_1^B\right)\overline{u}_1 = 0 \tag{2.11}$$

*or Principle of Virtual Displacements



Hence,

$$\int_{^tV} \left({^t\tau_{11,1}} + {^tf_1^B} \right) \overline{u}_1 \, d\,{^tV} = 0 \tag{2.12}$$

$$\underbrace{{^t\tau_{11}}\overline{u}_1 \Big|_{^tS_u}^{^tS_f}}_{\overline{u}_1^{S_f}\,{^t\tau_{11}}\,{^tS_f}} - \int_{^tV} \underbrace{\overline{u}_{1,1}}_{_t\overline{e}_{11}} {^t\tau_{11}} \, d^tV + \int_{^tV} \overline{u}_1 \, {^tf_1^B} \, d\,{^tV} = 0 \tag{2.13}$$

where ${^t\tau_{11}}|_{^tS_f} = {^tP_r}$.

Therefore we have

$$\int_{^tV} {_t\overline{e}_{11}} \, {^t\tau_{11}} \, d\,{^tV} = \int_{^tV} \overline{u}_1 \, {^tf_1^B} \, d\,{^tV} + \overline{u}_1^{S_f} \, {^tP_r} \, {^tS_f} \tag{2.14}$$

From (2.12) to (2.14) we simply used mathematics. Hence, if (2.2) and (2.3) are satisfied, then (2.14) must hold. If (2.14) holds, then also (2.2) and (2.3) hold!

Namely, from (2.14)

$$\int_{^tV} \overline{u}_{1,1} {^t\tau_{11}} d^tV = \overline{u}_1 {^t\tau_{11}} \Big|_{^tS_u}^{^tS_f} - \int_{^tV} \overline{u}_1 {^t\tau_{11,1}} d^tV = \int_{^tV} \overline{u}_1 {^tf_1^B} d^tV + \overline{u}_1^{S_f} {^tP_r} {^tS_f} \tag{2.15}$$

or

$$\int_{^tV} \overline{u}_1 \left({^t\tau_{11,1}} + {^tf_1^B} \right) d^tV + \overline{u}_1^{S_f} \left({^tP_r} - {^t\tau_{11}} \right) {^tS_f} = 0 \tag{2.16}$$

Now let $\overline{u}_1 = x \left(1 - \frac{x}{^tL} \right) \left({^t\tau_{11,1}} + {^tf_1^B} \right)$, where ${^tL}$ = length of bar.

Hence we must have from (2.16)

$${^t\tau_{11,1}} + {^tf_1^B} = 0 \tag{2.17}$$

and then also

$${^tP_r} = {^t\tau_{11}} \tag{2.18}$$